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V. *Some Problems of Atomic Stability.* *By* Prof. J. W. Nicholson, *M.A., D.Sc., F.R.S.*

Received Aug. 21, 1917.

It is well known that Sir J. J. Thomson's atomic model, in which the electrons are attracted to the centre of the atomic nucleus according to the law of direct distance, admits stable arrangements in which the electrons are not confined to the same plane. The Rutherford model, on the other hand, appears to be incompatible with such arrangements.* Certain special cases, however, are at first sight exceptional, and require a further examination. They are frequently used in the qualitative construction of atoms designed to have special physical properties. An atom of a pyramidal form, for example, consisting of a nucleus, a ring of electrons in the form of a circle whose axis goes through the nucleus, and a single stationary electron on this axis, is a simple case of a type which has been used to a great extent by Stark in his theoretical molecules, in which the linkage binding the atoms together consists of a stationary electron. The discussion of such pyramidal forms is the object of this Paper, in which their dynamical stability is investigated. It is to be pointed out that the investigation proceeds according to the classical dynamical method of treating problems of stability of moving systems, and has no relation to the criterion of stability used in the non-Newtonian mechanics of the atom developed, for example, by Bohr. While this new development is on its trial, it is nevertheless necessary to know where any suggested model stands with reference to the classical dynamics, and some of the constructions suggested by Stark and others have never been examined in any strict quantitative sense. Many are shown to be impossible whichever view of atomic dynamics is adopted.

Atoms of this type appear at first sight to have a possible existence with the solitary electron at rest, and the ring rotating uniformly about the axis, thereby constituting an exception to the statement that all the electrons in a Rutherford model must, according to the principles of Newtonian dynamics, be in the same ring. They are also of considerable interest on account of their unsymmetrical nature, which would

* "Phil. Mag.," April, 1914, p. 546; July, 1914, p. 94.

involve special properties for one particular electron. It is known that four electrons arranged at the corners of a tetrahedron can revolve in stable steady motion in a Thomson atom, and an analogous possibility might be expected in the present case. So far as radiation is concerned, these pyramidal atoms present no difficulty, for in steady motion of the ring the vector sum of the accelerations of all the charges in the atom is zero, and this is the condition for relatively small radiation.

The most fundamental conclusion reached in this paper is that no positively charged or neutral atom can exist in this form. Quite apart from any question of stability, it is not even possible, as may be shown rigorously, to satisfy the ordinary conditions for a steady rotation of the ring in such atoms.

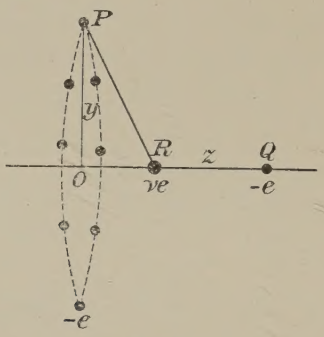

FIG. 1.

PO is the radius $=y$.
P is any electron.
Q is the solitary electron.
R is the nucleus.
RQ is the atomic axis.

This conclusion appears to remove the possibility of molecular structures of the type already mentioned. Moreover, even though an atom may theoretically take up an electron on its axis, if it is neutral or negatively charged before the operation, the ensuing structure is unstable as regards some of its more important vibrations. Its stability in fact is not comparable with that of a single ring, and it could not be endowed with any permanence. The vibrations which it can sustain, however, during a transitory existence, are of considerable interest and several numerical cases are worked out in detail. The direct application of Earnshaw's theorem of stability is not very helpful in atomic theory, in that unstable modes are, in general, those least readily excited.

We shall investigate, in the first place, a general pyramidal atom, in which there are n electrons in the base forming a ring and a single electron at the vertex, the nucleus νe being situated on the axis. Let y be the radius of the ring, x its distance from the nucleus, z the distance of the solitary electron from the nucleus, and m the mass of an electron. The orientation of an electron in the plane of the ring may be denoted by θ. The arrangement is indicated in the annexed figure.

The Lagrangian method supplies the readiest means of solution for the principal vibrations. The kinetic and potential energies are given by

$$2\tau = m\dot{z}^2 + mn(\dot{x}^2 + \dot{y}^2 + y^2\dot{\theta}^2),$$

$$V = -\frac{\nu e^2}{z} + \frac{ne^2}{\sqrt{y^2+(x+z)^2}} + \frac{ne^2 S_n}{4y} - \frac{\nu ne^2}{\sqrt{x^2+y^2}},$$

when the displacements are such as to retain a plane ring—the important case corresponding to vibrations of "class zero," as defined in previous investigations.* S_n has the usual meaning,

$$S_n = \sum_{r=1}^{n-1} \operatorname{cosec} \frac{r\pi}{n}.$$

The equations of motion become

$$m\ddot{z} = -\frac{\nu e^2}{z^2} + \frac{ne^2(x+z)}{\{y^2+(x+z)^2\}^{\frac{3}{2}}},$$

$$m\ddot{x} = -\frac{\nu e^2 x}{(x^2+y^2)^{\frac{3}{2}}} + \frac{e^2(x+z)}{[y^2+(x+z)^2]^{\frac{3}{2}}},$$

$$\frac{d}{dt}(\dot{y}^2\theta) = 0.$$

$$m(\ddot{y} - y\dot{\theta}^2) = \frac{e^2 S_n}{4y^2} - \frac{\nu e^2 y}{(x^2+y^2)^{\frac{3}{2}}} + \frac{e^2 y}{[y^2+(x+z)^2]^{\frac{3}{2}}}.$$

We have regarded the nucleus as fixed during the vibration. The knowledge of its effect on periods, already possessed, shows that it cannot alter the character of the solution, or even the numerical values to more than a few tenth metres, in the case of wave-lengths of emitted radiation. The error is in fact of order m/M, where M is the mass of the nucleus.

* "Monthly Notices" of R.A.S., Nov., 1911, *et seq.*

The equations of steady motion, derived by equating accelerations to zero, are reduced to

$$(x+z)z^2=\frac{\nu}{n}[y^2+(x+z)^2]^{\frac{3}{2}},$$

$$(x^2+y^2)^{\frac{3}{2}}=nxz^2,$$

$$my\omega^2=-\frac{e^2S_n}{4y^2}+\frac{\nu e^2y}{(x^2+y^2)^{\frac{3}{2}}}-\frac{e^2y}{[y^2+(x+z)^2]^{\frac{3}{2}}}.$$

The first pair serve to determine the ratios $x:y:z$, and do not involve ω, the steady angular velocity. Thus the structure of such an atom would be definite, in the geometrical sense, and a change in the angular velocity would only cause an expansion or contraction of all its distances in the same ratio. The relative distances of the atom in fact are determined solely by n and ν.

For the examination of the vibrations about steady motion, we may write $(x+x_1\ y+y_1\ z+z_1\ \omega+\varphi)$ for $(x,\ y,\ z,\ \theta)$, so that (x, y, z) now denote steady motion values, and the squares of x_1, y_1, z_1, and φ may be neglected in the usual manner. Then

$$\ddot{mz}_1=-V_{zx}x_1-V_{zy}y_1-V_{zz}z_1,$$

$$mn\ddot{x}_1=-V_{xx}x_1-V_{xy}y_1-V_{xz}z_1,$$

$$mn(\ddot{y}_1-\omega^2y_1-2\omega y\varphi)=-V_{xy}x_1-V_{yy}y_1-V_{yz}z_1,$$

$$y^2\varphi+2\omega y y_1=0,$$

the suffixes to V denoting differentiations. Thus, if all the variations are proportional to e^{iqt}, in a periodic variation from the steady motion,

$$mq^2z_1=V_{zx}x_1+V_{zy}y_1+V_{zz}z_1,$$

$$mnq^2x_1=V_{xx}x_1+V_{xy}y_1+V_{xz}z_1,$$

$$mn(q^2-3\omega^2)y_1=V_{xy}x_1+V_{y_1}y_1+V_{yz}z_1,$$

with the period equation

$$\begin{vmatrix} q^2-V_{xx}/mn & -V_{xy}/mn & -V_{xz}/mn \\ -V_{xy}/mn & q^2-3\omega^2-V_{yy}/mn & -V_{yz}/mn \\ -V_{xz}/mn & -V_{yz}/mn & \frac{q^2}{n}-V_{zz}/mn \end{vmatrix}=0,$$

which is a cubic. There are three principal vibrations of class zero, and the three values of q^2 must all be real and positive

if the stability of the system is to be comparable with that of a simple ring, which, moving as a whole in vibrations of class zero, is completely stable.

Let $$x/y=\alpha, \quad z/y=\beta.$$

Then the conditions of steady motion become

$$(1+\alpha^2)^{\frac{3}{2}}=n\alpha\beta^2,$$

$$\nu[1+(\alpha+\beta)^2]^{\frac{3}{2}}=n\beta^2(\alpha+\beta),$$

$$my^3\omega^2/e^2=-\frac{1}{4}S_n+\frac{\nu}{(1+\alpha^2)^{\frac{3}{2}}}-\frac{1}{[1+(\alpha+\beta)^2]^{\frac{3}{2}}}$$

$$=-\frac{1}{4}S_n+\frac{\nu}{n\alpha\beta^2}-\frac{\nu}{n\beta^2(\alpha+\beta)}$$

$$=-\frac{1}{4}S_n+\frac{\nu}{n\alpha\beta(\alpha+\beta)},$$

substituting from the first two conditions in order to satisfy the third. This third condition relates the angular velocity and radius of the ring in the usual manner, and the first two conditions serve to determine the shape of the atom.

We find the following values for the derivates of V :—

$$-\frac{1}{mn}V_{yz}=-\frac{3\nu e^2}{mny^3}\cdot\frac{\alpha+\beta}{[1+(\alpha+\beta)^2]^{\frac{5}{2}}} \quad \ldots\ldots\ldots \quad (1)$$

$$=-\frac{3\nu e^2}{mn^2y^3\beta^2}\cdot\frac{1}{1+(\alpha+\beta)^2}$$

$$-\frac{1}{mn}V_{zx}=\frac{e^2\nu^2}{mn^2y^3\beta^2(\alpha+\beta)}\cdot\frac{1-2(\alpha+\beta)^2}{1+(\alpha+\beta)^2} \quad \ldots\ldots \quad (2)$$

$$-\frac{1}{mn}V_{zz}=\frac{\nu e^2}{mny^3\beta^3}\left\{2+\frac{\beta}{\alpha+\beta}\cdot\frac{1-2(\alpha+\beta)^2}{1+(\alpha+\beta)^2}\right\} \quad \ldots\ldots \quad (3)$$

$$-\frac{1}{mn}V_{xy}=\frac{3\nu e^2}{mny^3\beta^2}\left\{\frac{1}{1+\alpha^2}-\frac{1}{1+(\alpha+\beta)^2}\right\} \quad \ldots\ldots \quad (4)$$

$$-\frac{1}{mn}V_{xx}=\frac{\nu e^2}{mny^3\beta^2}\left\{\frac{1-2(\alpha+\beta)^2}{(\alpha+\beta)\{1+(\alpha+\beta)^2\}}-\frac{1-2\alpha^2}{\alpha(1+\alpha^2)}\right\} \quad \ldots \quad (5)$$

$$-\frac{V_{yy}}{mn}=\frac{e^2}{my^3}\left\{\frac{1}{2}S_n-\frac{\nu(\alpha^2-2)}{n\alpha\beta^2(1+\alpha^2)}+\frac{\nu}{n\beta^2(\alpha+\beta)}\frac{(\alpha+\beta)^2-2}{1+(\alpha+\beta)^2}\right\} \quad \ldots \quad (6)$$

from which the period equation may be expressed in general terms as an equation in q/ω with numerical co-efficients, when (α, β) are found.

The Possible Steady Motions.

The ratios (α, β) in general satisfy

$$(1+\alpha^2)^{\frac{3}{2}}=n\alpha\beta^2,$$
$$\nu[1+(\alpha+\beta)^2]^{\frac{3}{2}}=n(\alpha+\beta)\beta^2.$$

Write $\alpha=\tan\theta$, so that

$$\beta^2=\frac{1}{n\sin\theta\cos^2\theta}, \quad \beta=\frac{1}{\cos\theta\sqrt{n\sin\theta}}.$$

Therefore θ satisfies

$$\left\{1+\left(\tan\theta+\frac{\sec\theta}{\sqrt{n\sin\theta}}\right)^2\right\}^3=\frac{n^2}{\nu^2}\cdot\frac{1}{n^2\sin^2\theta\cos^4\theta}\left(\tan\theta+\frac{\sec\theta}{\sqrt{n\sin\theta}}\right)^2,$$

or, on reduction,

$$[n\sin\theta\cos^2\theta+(1+\sqrt{n}\sin^{\frac{3}{2}}\theta)^2]^3=\frac{n^2}{\nu^2}(1+\sqrt{n}\sin^{\frac{3}{2}}\theta)^2.$$

Since by their nature, α and β are positive, the necessary value of θ must lie between 0 and $\frac{\pi}{2}$. It is evident that when n is less than ν, no such value of θ can satisfy the equation, so that *a pyramidal form of a positively charged atom is not possible.* The same result applies to a neutral atom, with $n=\nu-1$.

When the atom has a single negative charge, $n=\nu$, and the solution is $\theta=0$, or $\alpha=0$, $\beta=\infty$, indicating the single-ring type of a neutral atom, since the extra electron is at an infinite distance. But real values of α and β are possible for an atom with at least one negative charge. We shall work out in detail the case of a nucleus of strength $5e$, with seven electrons, six of them being in a ring.

Writing $n\sin\theta=w^2$, the new magnitude w is determined by

$$\left[w^2\left(1-\frac{w^4}{n^2}\right)+\left(1+\frac{w^3}{n}\right)^2\right]^3=\frac{n^2}{\nu^2}\left(1+\frac{w^3}{n}\right)^2,$$

or

$$\left(1+w^2+2\frac{w^3}{n}\right)^3=\frac{n^2}{\nu^2}\left(1+\frac{w^3}{n}\right)^2,$$

where, in terms of w,

$$\alpha=w^2/(n^2-w^4)^{\frac{1}{2}}, \quad \beta=n/w(n^2-w^4)^{\frac{1}{2}}.$$

In the case of the suggested illustration, $n=6$, $\nu=5$, and

$$\left(1+w^2+\frac{1}{3}w^3\right)^3=\frac{16}{25}\left(1+\frac{w^3}{6}\right)^2.$$

This equation has a root between zero and unity, but no other real root. Thus, as would be expected, only one such arrangement is possible. The single-ring arrangement corresponds to a neglected factor $w=0$.

The equation can be solved by continued approximation, and the value of the real root, to five significant figures, is

$$w=0{\cdot}34721,$$

leading to

$$\alpha=0{\cdot}020096,$$
$$\beta=2{\cdot}8807, \qquad \alpha+\beta=2{\cdot}9008,$$

and thence,

$$\omega^2=3{\cdot}1349\frac{e^2}{my^3},$$

showing that a steady motion is possible. The following are then obtained after some reduction :

$$V_{yz}=0{\cdot}008506mn\omega^2, \qquad V_{zx}=0{\cdot}015472mn\omega^2,$$
$$V_{zz}=-0{\cdot}003674mn\omega^2, \qquad V_{xy}=-0{\cdot}08585mn\omega^2,$$
$$V_{xx}=1{\cdot}6112mn\omega^2, \qquad V_{yy}=-4{\cdot}3599mn\omega^2,$$

and the period equation is

$$\begin{vmatrix} q^2/\omega^2-1{\cdot}6112 & 0{\cdot}08585 & -0{\cdot}01547 \\ 0{\cdot}08585 & q^2/\omega^2+1{\cdot}3599 & -0{\cdot}00851 \\ -0{\cdot}01547 & -0{\cdot}00851 & q^2/\omega^2+0{\cdot}00367 \end{vmatrix}=0,$$

or on expansion, if $\rho=q^2/\omega^2$,

$$\rho^3-0{\cdot}2293\rho^2-2{\cdot}20584\rho-0{\cdot}04974=0.$$

This equation has a positive root greater than unity, and two negative roots indicating unstable modes. The vibrations now investigated are those of class zero—the simplest vibrations, and those set up most readily—and the fact that two are unstable shows that the atom could only have a very transitory existence.

The positive root is found to be

$$q^2/\omega^2=1{\cdot}6145,$$

or

$$q/\omega=1{\cdot}2706,$$

which, of course, needs a small correction, arising from the fact that the positive nucleus of the atom moves slightly during a vibration.

We may notice that the unstable modes have no periodic feature, for q/ω is in each of them purely imaginary, and the

corresponding motion in these modes is a continued enlargement of the atom in all its dimensions, whose ratios remain constant.

The Energy of a Pyramidal System.

At this point we must notice an important formula for the energy of a system of the present type. It is, in fact, a special case of a much more general theorem relating to systems of charges in steady motion under the inverse square law. We do not, however, in the present Paper, discuss the more general significance of the energy relation, as it is not directly relevant. The energy of the present system may be written in the form

$$T+V=\frac{1}{2}mny^2\omega^2+\frac{e^2}{y}\left\{-\frac{\nu}{\beta}+\frac{n}{[1+(\alpha+\beta)^2]^{\frac{1}{2}}}+\frac{1}{4}nS_n-\frac{n\nu}{(1+\alpha^2)^{\frac{1}{2}}}\right\},$$

or by virtue of the equations of steady motion,

$$\begin{aligned}T+V&=\frac{1}{2}mny^2\omega^2+\frac{e^2}{y}\left\{-\frac{\nu}{\beta}+\frac{\nu}{\beta^2}\cdot\frac{1+(\alpha+\beta)^2}{\alpha+\beta}+\frac{nS_n}{4}-\frac{\nu}{\alpha\beta^2}(1+\alpha^2)\right\}\\&=\frac{1}{2}mny^2\omega^2+\frac{e^2}{y}\left\{\frac{1}{4}nS_n+\frac{\nu}{\beta^2}\left(\frac{1}{\alpha+\beta}-\frac{1}{\alpha}\right)\right\}\\&=\frac{1}{2}mny^2\omega^2-\frac{ne^2}{y}\left\{-\frac{1}{4}S_n-\frac{\nu}{n\alpha\beta(\alpha+\beta)}\right\}\\&=-\frac{1}{2}mny^2\omega^2.\end{aligned}$$

To this a constant C must be added, being the energy of the system when scattered to a state of infinite dispersion. It is remarkable that the formula for the energy is of the same type as for a single ring, in terms of radius and angular velocity.

Thus for n electrons in a ring of radius a, the energy is

$$C-\frac{1}{2}mna^2\omega^2,$$

and the same formula is valid for the total energy if there is also a stationary electron on the axis, and, as will appear later, if there are two such stationary electrons, one on either side of the axis, thereby forming a symmetrical atom.

A Symmetrical Atom with Two Stationary Electrons.

In order to complete the investigation, one further problem must be considered, relating to an atom which at first sight also appears to contain a greater possibility of limited stability, combined with symmetry.

We therefore next take up the case, of which the analysis is simpler, of a ring of n electrons with a solitary electron on *both* sides.

Attention may be restricted to the case in which the electrons are always equidistant from the nucleus, so that the ring does not vibrate transversely. If y is its radius and x

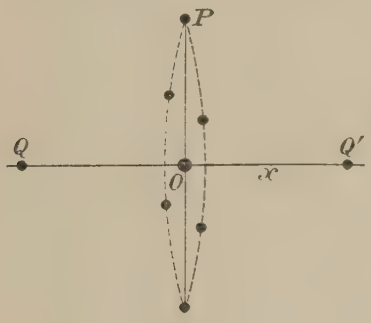

FIG. 2.

O is the nucleus.	PO=radius=y.
Q, Q' the solitary electrons.	OQ=OQ'=x.
P an electron of the ring.	

the distance in the figure, the Lagrangian function for such vibrations is given by

$$2T=mn(\dot{y}^2+y^2\dot{\theta}^2)+2m\dot{x}^2,$$

$$V=-\left(2\nu-\frac{1}{2}\right)\frac{e^2}{x}+\frac{2ne^2}{\sqrt{x^2+y^2}}-\frac{ne^2}{y}\left(\nu-\frac{1}{2}S_n\right),$$

and

$$2m\ddot{x}=-\frac{\partial V}{\partial x}$$

$$mn(\ddot{y}-y\dot{\theta}^2)=-\frac{\partial V}{\partial y}$$

$$\frac{d}{dt}(y^2\theta)=0.$$

The conditions of steady motion are

$$\nu-\frac{1}{4}=\frac{nx^3}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\omega^2=\frac{e^2}{my^3}\left\{\nu-\frac{1}{4}S_n-\frac{2y^3}{(x^2+y^2)^{\frac{3}{2}}}\right\}.$$

For the small vibration, writing as before

$$\theta=\omega+\varphi, \quad \text{and} \quad x+x_1, \quad y+y_1, \quad \text{for } (x, y),$$

so that (x, y) are the steady motion values,

$$2m\ddot{x}_1=-x_1V_{xx}-y_1V_{xy},$$

$$mn(\ddot{y}_1+3\omega^2y_1)=-x_1V_{xy}-y_1V_{yy},$$

or if the displacements are proportional to e^{iqt},

$$x_1\left(q^2-\frac{V_{xx}}{2m}\right)=y_1\frac{V_{xy}}{2m}$$

$$y_1\left(q^2-3\omega^2-\frac{V_{yy}}{mn}\right)=x_1\frac{V_{xy}}{mn},$$

giving a period equation

$$\left(q^2-\frac{V_{xx}}{2m}\right)\left(q^2-3\omega^2-\frac{V_{yy}}{mn}\right)=\frac{1}{2n}\left(\frac{V_{xy}}{m}\right)^2,$$

with two principal modes of vibration. We find

$$V_{xx}=-(4\nu-1)\frac{e^2}{x^3}-\frac{2ne^2(y^2-2x^2)}{(x^2+y^2)^{\frac{5}{2}}}=\frac{-6ne^2y^2}{(x^2+y^2)^{\frac{5}{2}}},$$

$$V_{yy}=-\frac{2ne^2}{y^3}\left(\nu-\frac{1}{4}S_n\right)-\frac{2ne^2(x^2-2y^2)}{(x^2+y^2)^{\frac{5}{2}}}=-2nm\omega^2-\frac{6ne^2x^2}{(x^2+y^2)^{\frac{5}{2}}},$$

$$V_{xy}=\frac{6ne^2xy}{(x^2+y^2)^{\frac{5}{2}}}.$$

Let $$\sigma=\frac{x}{y}=\left[\left(\frac{\nu}{n}-\frac{1}{4n}\right)^{-\frac{2}{3}}1-\right]^{-\frac{1}{2}}.$$

Then

$$\omega^2=\frac{e^2}{my^3}\left\{\nu-\frac{1}{4}S_n-\frac{2}{(1+\sigma^2)^{\frac{3}{2}}}\right\},$$

$$\frac{V_{xx}}{2m}=-\frac{e^2}{my^3}\left\{\left(2\nu-\frac{1}{2}\right)\frac{1}{\sigma^3}+\frac{n(1-2\sigma^2)}{(1+\sigma^2)^{\frac{5}{2}}}\right\}=-\frac{3n}{(1+\sigma^2)^{\frac{5}{2}}}\frac{e^2}{my^3},$$

$$\frac{V_{yy}}{mn}=-\frac{e^2}{my^3}\left\{2\left(\nu-\frac{1}{4}S_n\right)+\frac{2(\sigma^2-2)}{(1+\sigma^2)^{\frac{3}{2}}}\right\}=-2\omega^2-\frac{6\sigma^2}{(1+\sigma^2)^{\frac{5}{2}}}\frac{e^2}{my^3},$$

$$\frac{V_{xy}}{m}=\frac{e^2}{my^3}\left\{\frac{6n\sigma}{(1+\sigma^2)^{\frac{5}{2}}}\right\}.$$

In this type of vibration the positive nucleus does not move and calculations of vibrations for this system are free from the unusual complexity of the former. It is evident that no steady motion exists for atoms of the present type, if they have a positive charge.

We shall calculate the case $\nu=5$, $n=6$. Thus

$$\sigma=[(0{\cdot}79167)^{-\frac{2}{3}}-1]^{-\frac{1}{2}}=(0{\cdot}16852)^{-\frac{1}{2}}=2{\cdot}4359$$

$$(1+\sigma^2)^{-\frac{2}{3}}=0{\cdot}054770,$$

$$\omega^2=\frac{e^2}{my^3}(3{\cdot}0631),$$

$$\frac{V_{xx}}{2m}=-\omega^2(0{\cdot}046418),$$

$$\frac{V_{yy}}{mn}=-\omega^2(2{\cdot}091812),$$

$$\frac{V_{xy}}{m}=\omega^2(0{\cdot}22614), \quad \frac{1}{2n}\left(\frac{V_{xy}}{m}\right)^2=0{\cdot}004262,$$

and the period equation is

$$\left(\frac{q^2}{\omega^2}+0{\cdot}04642\right)\left(\frac{q^2}{\omega^2}-0{\cdot}90819\right)=0{\cdot}004262,$$

with a positive and a negative root. The negative root does not correspond to a periodic disturbance, and the system is unstable. The value of the positive root is ultimately

$$q^2/\omega^2=0{\cdot}91263,$$

or

$$q/\omega=0{\cdot}95532.$$

The 7-ring with Two Solitary Electrons.

Before commenting further on the vibrations peculiar to the system last dealt with, we proceed to collect numerical data of sufficient completeness to make possible a general view of these systems. For this purpose we perform a similar calculation with seven electrons in the ring. The stages of the calculation are set forth below, with $n=7$, $\nu=5$.

$$\sigma=\left[\left(\frac{\nu}{n}-\frac{1}{4n}\right)^{-\frac{2}{3}}-1\right]^{-\frac{1}{2}}=(0{\cdot}29500)^{-\frac{1}{2}}=1{\cdot}84115,$$

$$1+\sigma^2=4{\cdot}38983,$$

$$\omega^2=\frac{e^2}{my^3}\left(2{\cdot}6952-\frac{2}{(1+\sigma^2)^{\frac{1}{2}}}\right)=\frac{e^2}{my^3}(2{\cdot}4778),$$

$$\frac{V_{xx}}{2m}=-\omega^2(0{\cdot}20991),$$

$$\frac{V_{yy}}{2m}=-\omega^2(2{\cdot}203305),$$

$$\frac{V_{xy}}{m}=\omega^2(0{\cdot}77295), \quad \frac{1}{2n}\left(\frac{V_{xy}}{m}\right)^2=0{\cdot}042675,$$

and the period equation is, with $\rho = q^2/\omega^2$,

$$(\rho + 0{\cdot}20991)(\rho - 0{\cdot}796695) = 0{\cdot}042675,$$

or $$\rho^2 - 0{\cdot}58678\ \rho - 0{\cdot}20991 = 0\ ;$$

whence $$\rho = 0{\cdot}83743;$$

or $$q/\omega = 0{\cdot}9151$$

for the real root. There is an imaginary root as before.

The 8-Ring with Two Solitary Electrons.

In this case,

$$\sigma = [(0{\cdot}59375)^{-\frac{3}{2}} - 1]^{-\frac{1}{2}} = (0{\cdot}41557)^{-\frac{1}{2}} = 1{\cdot}55123,$$

$$1 + \sigma^2 = 3{\cdot}4063,$$

$$\omega^2 = \frac{e^2}{my^3}\left(2{\cdot}1951 - \frac{2}{(1+\sigma^2)^{\frac{3}{2}}}\right) = \frac{e^2}{my^3}(1{\cdot}8770),$$

$$\frac{V_{xx}}{2m} = -\omega^2(0{\cdot}59709),$$

$$\frac{V_{yy}}{mn} = -\omega^2(2{\cdot}35920),$$

$$\frac{V_{xy}}{m} = \omega^2(1{\cdot}8525), \quad \frac{1}{2n}\left(\frac{V_{xy}}{m}\right)^2 = 0{\cdot}214475,$$

and if $$\rho = q^2/\omega^2,$$

$$(\rho + 0{\cdot}59709)(\rho - 0{\cdot}64080) = 0{\cdot}21447,$$

or $$\rho^2 - 0{\cdot}04371\ \rho - 0{\cdot}59709 = 0,$$

with a positive root

$$\rho = 0{\cdot}79490,$$

or $$q/\omega = 0{\cdot}8916,$$

and the usual unstable vibration.

The Energy of the Atom.

We can show that the energy of an atom with a stationary electron on either side of the ring is still given by the same type of formula, as with an electron on one side only, or with a single ring only. For the energy of the atom with two stationary electrons, and n rotating electrons in the ring is, with the previous notation,

$$T + V = \tfrac{1}{2}mn\omega^2 y^2 + \frac{e^2}{y}\left\{\frac{1}{2\sigma} - n\left(\nu - \frac{1}{4}S_n\right) + \frac{2n}{\sqrt{1+\sigma^2}} - \frac{2\nu}{\sigma}\right\},$$

where

$$\nu - \frac{1}{4} = n\sigma^3/(1+\sigma^2)^{\frac{3}{2}}.$$

Thus

$$T+V=\frac{1}{2}mn\omega^2y^2+\frac{e^2}{y}\left\{\frac{1}{2\sigma}-n\left(\nu-\frac{1}{4}S_n\right)-\frac{2\nu}{\sigma}+\frac{2}{\sigma^3}\left(\nu-\frac{1}{4}\right)(1+\sigma^2)\right\},$$

$$=\frac{1}{2}mn\omega^2y^2+\frac{e^2}{y}\left\{\frac{2}{\sigma^3}\left(\nu-\frac{1}{4}\right)-n\left(\nu-\frac{1}{4}S_n\right)\right\},$$

$$=\frac{1}{2}mn\omega^2y^2-\frac{ne^2}{y}\left\{\nu-\frac{1}{4}S_n-\frac{2}{(1+\sigma^2)^{\frac{3}{2}}}\right\},$$

$$=-\frac{1}{2}mn\omega^2y^2,$$

by the conditions of steady motion. We add the constant C dependent on the zero of measurement, and obtain the energy

as $$C-\frac{1}{2}mn\omega^2y^2,$$

a formula again of the same type as for the single ring of radius y and angular velocity ω.

If the stable periods actually are periods of emitted radiation, and if radiation takes place in quanta associated with the angular momentum of the ring, it is known that the principal wave-lengths to which they give rise form a series with constant difference of (wave-length)$^{\frac{1}{2}}$, in the case of a single ring. Such series have been detected in the solar coronal spectrum.* If a system of the present kind is also subject to such laws, its stable vibrations would give a spectrum of the same type: but the greater instability of the system would probably preclude its detection.

Transverse Vibrations of the Ring.

It is evident that the vibrations already dealt with in the atoms with two solitary electrons correspond to the simplest vibration in the plane, in the case of a single ring. For the rings of these atoms have only moved in their own planes hitherto. It is necessary to determine the vibration, also of class zero, which corresponds to the simplest transverse vibration of a simple-ring atom—in this case a vibration in which the ring moves transversely, while the displacements

* "Monthly Notices" of R.A.S., Supp. No. 1912, p. 730, *et seq.*

of all the electrons of the ring are equal at any instant. If the ring is displaced by an amount s, as in the figure, and (x, z) are the distances of the solitary electrons from the nucleus, at present regarded as stationary, we may write

$$2T=mn(\dot{y}^2+y^2\dot{\theta}^2+\dot{s}^2)+m(\dot{x}^2+\dot{z}^2),$$

$$V=-ve^2\left(\frac{1}{x}+\frac{1}{z}\right)+\frac{e^2}{x+z}+ne^2\left\{\frac{1}{\sqrt{y^2+(x-s)^2}}+\frac{1}{\sqrt{y^2+(z+s)^2}}\right\}$$

$$+\frac{ne^2s_n}{4y}-\frac{ne^2}{\sqrt{y^2+s^2}},$$

where in the steady motion, $x=z$, $s=0$.

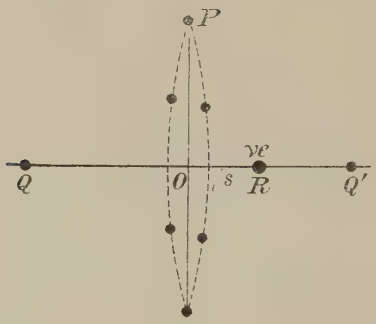

Fig. 3.

$OQ=x.$ | $OR=s.$
$OQ'=z=x-s.$ | $OP=\text{radius}=y.$

The equations of motion for the co-ordinates (x, z, s) are

$$m\ddot{x}=-V_x, \quad m\ddot{z}=-V_z, \quad m\ddot{s}=-V_x,$$

or if $x+x_1$, $z+z_1$ are put for (x, z), which now denote steady motion values, we derive

$$mq^2x_1=x_1V_{xx}+y_1V_{xy}+z_1V_{xz}+s_1V_{xs},$$
$$mq^2y_1=x_1V_{xy}+y_1V_{yy}+z_1V_{yz}+s_1V_{ys},$$
$$mq^2s_1=x_1V_{sx}+y_1V_{sy}+z_1V_{sz}+s_1V_{ss},$$

where $x=z$, $s=0$ after differentiation.

There are relations among these coefficients. For example, V_{sy} is given by

$$V_{sy}=\frac{\partial}{\partial y}\left\{\frac{ne^2s}{(y^2+s^2)^{\frac{3}{2}}}-\frac{(s-x)ne^2}{[y^2+(s-x)^2]^{\frac{3}{2}}}-\frac{(s+z)ne^2}{(y^2+(z+s)^2)^{\frac{3}{2}}}\right\},$$

$$=\frac{\partial}{\partial y}\{0\}=0,$$

when $s=0$, $x=z$.

The last equation is therefore independent of y_1.

Moreover, when s vanishes, V is symmetrical in x and z, and

$$V_{xx}=V_{zz}.$$

We can also show that

$$V_{xs}=-V_{zs}, \quad V_{ss}=-2V_{xs}, \quad V_{xy}=V_{zy},$$

after steady motion values are substituted. The equations, therefore, take the form

$$x_1(mq^2-V_{xx})=y_1V_{xy}+z_1V_{xz}+s_1V_{xs},$$

$$z_1(mq^2-V_{xx})=y_1V_{xy}+x_1V_{xz}-s_1V_{xs},$$

$$s_1(mnq^2+2V_{xs})=(x_1-z_1)V_{xs},$$

or, with the elimination of y_1,

$$(x_1-z_1)(mq^2-V_{xx}+V_{xz})=2s_1V_{xs},$$

$$(x_1-z_1)V_{xs}=s_1(mnq^2+2V_{xs}),$$

with the solution

$$x_1=z_1, \quad s_1=0$$

(which is the vibration already discussed), or

$$(mq^2-V_{xx}+V_{xz})(mnq^2+2V_{xs})=2(V_{xs})^2.$$

This is the period equation of the transverse vibrations of class zero, corresponding to the simple transverse vibrations of a single ring.

With steady motion values,

$$V_{xz}=\frac{e^2}{4x^3}, \qquad V_{xx}=-\frac{2ne^2}{x^3}+\frac{e^2}{4x^3}-ne^2\left\{\frac{1}{(x^2+y^2)^{\frac{3}{2}}}-\frac{3x^2}{(x^2+y^2)^{\frac{5}{2}}}\right\},$$

$$V_{xs}=ne^2\left\{\frac{1}{(x^2+y^2)^{\frac{3}{2}}}-\frac{3x^2}{(x^2+y^2)^{\frac{5}{2}}}\right\},$$

or with $x/y=\sigma$ as before,

$$\frac{V_{xs}}{mn}=\frac{e^2}{my^3}\left\{\frac{1}{(1+\sigma^2)^{\frac{3}{2}}}-\frac{3\sigma^2}{(1+\sigma^2)^{\frac{5}{2}}}\right\},$$

$$\frac{V_{xz}-V_{xx}}{m}=\frac{e^2}{my^3}\left\{\frac{2\nu}{\sigma^3}+\frac{n}{(1+\sigma^2)^{\frac{3}{2}}}-\frac{3n\sigma^2}{(1+\sigma^2)^{\frac{5}{2}}}\right\},$$

with the period equation

$$\left(q^2+\frac{V_{xz}-V_{xx}}{m}\right)\left(q^2+\frac{2V_{xs}}{mn}\right)=2n\left(\frac{V_{xs}}{mn}\right)^2.$$

Numerical Values.

Proceeding to the numerical calculation of the various systems, we obtain the following results :—

(1) For a ring of six electrons, with $\nu=5$,

$$\sigma=2\cdot4359, \qquad \omega^2=3\cdot0631\frac{e^2}{my^3},$$

$$V_{xs}=-0\cdot028026\ mn\omega^2,$$

$$V_{xz}-V_{xx}=0\cdot05771m\omega^2, \qquad 2n\left(\frac{V_{xs}}{mn}\right)^2=0\cdot009425,$$

$$(q^2/\omega^2+0\cdot05771)(q^2/\omega^2-0\cdot05605)=0\cdot009425,$$

with a root

$$q^2/\omega^2=0\cdot11212,$$

so that

$$q/\omega=0\cdot3348,$$

and a negative root.

(2) For a ring of seven electrons, with $\nu=5$,

$$\sigma=1\cdot84115, \qquad \omega^2=2\cdot4778\frac{e^2}{my^3},$$

$$V_{xs}=-0\cdot05717mn\omega^2,$$

$$V_{xz}-V_{xx}=0\cdot24225m\omega^2, \qquad 2n\left(\frac{V_{xs}}{mn}\right)^2=0\cdot04672,$$

$$(q^2/\omega^2+0\cdot24225)(q^2/\omega^2-0\cdot11554)=0\cdot04672,$$

$$q^2/\omega^2=0\cdot21722, \qquad q/\omega=0\cdot4661.$$

(3) For a ring of eight electrons, with $\nu=5$,

$$\sigma=1\cdot55123, \qquad \omega^2=1\cdot8770\frac{e^2}{my^3},$$

$$V_{xs}=-0\cdot09485mn\omega^2,$$

$$V_{xz}-V_{xx}=1\cdot5248m\omega^2, \quad 2n\left(\frac{V_{xs}}{mn}\right)^2=0\cdot14394,$$

$$(q^2/\omega^2+1\cdot5248)(q^2/\omega^2-0\cdot1897)=0\cdot14394,$$

$$q^2/\omega^2=0\cdot2699, \qquad q/\omega=0\cdot5195.$$

Collecting the results, the following vibrations are possible, with $\nu=5$:—

(1) $n=6$, $\omega^2=3{\cdot}0631\frac{e^2}{my^3}$, $q/\omega=0{\cdot}9553$, $0{\cdot}3348$.

(2) $n=7$, $\omega^2=2{\cdot}4778\frac{e^2}{my^3}$, $q/\omega=0{\cdot}9151$, $0{\cdot}4661$.

(3) $n=8$, $\omega^2=1{\cdot}8770\frac{e^2}{my^3}$, $q/\omega=0{\cdot}8916$, $0{\cdot}5195$.

And with them are associated several unstable modes of equal importance. The systems of this type are, therefore, essentially unstable to a degree quite transcending that of simple-ring atoms. They cannot well form part of any molecular structure, and without the necessity of more formal and general investigations, we may conclude that conceptions of molecules such as those of Stark cannot survive the test of quantitative treatment on an ordinary dynamical basis. They may nevertheless be possible under the rule of a non-Newtonian dynamics, but the criticism to which they are open, even in this case, is that, from a consideration of steady motions alone—in which the two varieties of dynamics are identical, as in Bohr's method—the individual atoms can only exist when negatively charged, so that no two of them could well form an electrically neutral molecule.

General Remarks.

Earnshaw's theorem regarding systems of electric charges indicates that atoms of the type contemplated in the present paper are unstable. But this fact has not precluded physicists from making use of them, just as the demonstrable fact that coplanar rings of electrons in an atom are mathematically impossible, is even now frequently ignored by speculators in atomic theory. In the present case, where not the steady motion of an atom, but its stability, is in question, this procedure on the part of physicists is, however, frequently justified. A single ring of electrons in an atom cannot be stable, but the disturbances which break up the atom on account of instability are those which are least likely to occur, so that in spite of this possibility, the atom can have as long an existence as an electrically neutral system, as is demanded by its known properties. Atoms of the type in this Paper, however, do not satisfy this criterion, and their existence could only be very transitory as negatively charged systems. In any other form,

or as components of a molecule, they cannot exist at all, apart from all question of stability, and this particular conclusion is not dependent on the system of dynamical reasoning adopted. It appears to vitiate entirely all molecular constructions such as those of Stark.

In the Paper some of the stable vibrations are worked out in special cases for negative systems which would, from their nuclear charge, be identical chemically with others known to exist in the solar corona. Although the full comparison is not given, it may be stated that the coronal spectrum gives no evidence of the existence of such systems. Their instability must be too great. Such a conclusion is in accordance with the very definite instability of their more important modes of vibration.

ABSTRACT.

The Paper is mainly concerned with the possible existence and stability of atoms, and of molecules formed after the manner suggested by Stark, the link between the atoms in a molecule being provided by a stationary electron on the molecular axis. Atoms on the Rutherford model, though dynamically unstable, are stable for the simple vibrations ordinarily excited; but it is shown in the Paper that atoms with such a stationary electron have a much more limited degree of stability. Moreover, they cannot exist even in an undisturbed state unless they are endowed with a negative charge, for no steady motion is possible, and this conclusion extends even to atoms regulated according to a dynamics such as that of Bohr. Stark's conclusions do not, therefore, survive a quantitative treatment, and molecules cannot be formed in the manner he supposes.

The Paper also discusses the more symmetrical problem, in which there are two such stationary electrons in an undisturbed atom, and it is shown that systems with a transitory existence, which are known by their spectra to occur in the solar corona, are apparently unaccompanied by the still more transitory systems which would be formed by the attachment of an electron after the manner of Stark. This is a further argument against the possibility that two atoms in a molecule can be linked by a single electron, or by two electrons, which attract both atoms.

DISCUSSION.

Mr. T. Smith asked if it would be possible to make some such generalisation as that symmetry in the system was essential to stability.

Dr. Borns asked what was meant by the statement that the atom with $+5e$ and $-7e$ is strongly represented in the solar corona.

Dr. R. S. Willows said it was of great advantage that eminent mathematicians were investigating the suggested models of the atom and attempting to compare the results with experiment. Stark had made some brilliant experimental discoveries, but his theoretical deductions appeared to be faulty.

Prof. Nicholson, in reply, said he thought some such generalisation as that suggested by Mr. Smith could be made. At any rate, the greater the symmetry of the system the more liable it is to be stable. In reply to Dr. Borns, he meant that certain lines of strong intensity in the spectrum of the corona were referable to such a system.

VI. *Presidential Address, delivered January 25, 1918, by Prof.* C. V. Boys, *F.R.S.*

The cruel exigencies of the war have seriously impeded the work of the Physical Society, as of all our scientific institutions. Many of our members are doing their duty to the country at the front, and many are devoting themselves to problems and hard work imposed by the war conditions, and but little time has been available for those scientific pursuits in which as members of the Physical Society we are normally engaged.

Since my predecessor delivered his address a notable event has occurred which has stirred the scientific community to a degree which I think has been unnecessary. I allude to the passing of the Daylight Saving Bill. Looked at scientifically the thing is ridiculous and a sham, and as such it is naturally hateful to us. But the community at large is not scientific. Thanks to our school education, the general public has a very vague idea of the meaning of time. It has a vague idea that the sun has something to do with the time of day, and that in a crude kind of way it may even be used to indicate the time, but that it is not so good as a clock. In the stress of war the public has been made to realise that it is desirable in the summer months for us to start the day sooner and save, not daylight, but paraffin and gas, and that the ultimate objects for which Mr. Willett strove were in most respects extremely advantageous. The practical and simple operation of putting all the clocks wrong, so avoiding alterations of postal, banking and other business hours, or of reprinting the railway time-tables, though hateful to us in principle, did not disturb the general public, not because all the clocks told the same lie, but because, so far as the multitude were concerned, it was not a lie at all; for, as I have said, they have the vaguest idea of the relation of the sun to time of day, for is it not morning before lunch and after noon after lunch? And they have no understanding at all, either of mean time, of Greenwich or local mean time, or of equation of time. And then when the question was discussed and settled in Parliament we may conclude that the gentlemen there aptly represent their constituents, for the member who introduced the Bill, a member of this Society also, being apparently of that opinion, made fun of the whole system of Greenwich mean time, so that they

should not do violence to their collective conscience in passing the Bill.

I do not propose to weary you with a discussion of the metric question, for all that can be said has been said so often, but it is of importance in relation to education. I believe the reason why in England we are at school so backward in mathematics is that the scanty time that is available is for so long not devoted to mathematics at all, but to the wholly unprofitable business of committing to memory tables of weights and measures, and learning all kinds of compound operations which the use of these medieval relics require. While agreeing with other representatives of science that the classical schoolmaster has too much influence in determining the course of our public school education, I do feel very strongly that so long as we can afford to allow our education to be something more than merely utilitarian, we should not give up the study of Latin and Greek, certainly not of both. Besides being the parent of the modern Latin tongues, which seem so well adapted for expressing clearly mathematical and scientific reasoning, Latin gives us a clean cut grammar from which the principles of grammar can be assimilated with a facility that is impossible with a modern language such as English, which has practically lost its grammar. I feel that much of the modern slovenly writing from which authors of scientific Papers are not wholly free, is the result of insufficient literary education. If we had less time more humanly spent we might get further in reading Latin authors, and in addition acquire some facility in writing Latin, and use it later as in the days of Newton as the language of general international communication, at least for our scientific publications.

Another matter of public importance which has relation to the meagre recognition of science as an element of general education has been so consistently urged that I do not care to repeat what we know. I would only say, mainly in answer to any prevalent ideas that our officials need not be scientific, because they can get all the scientific advice that they want, first, that possibly they may not always know when they want —*i.e.*, need—such advice, and, secondly, without scientific training of any kind they may not appreciate the meaning or force of it when they get it. I cannot help being reminded of a statement made by a fellow-student of the School of Mines that in his country there were some people who were so superior to ordinary folk that they would not allow their sons to be taught

how to write, as that was clerk's work, and they could get all the writing they wanted done for them without degrading themselves. This may or may not be true, but the parallelism is very close. There is, however, another parallel which I know to be true. When in the early days of the Physical Laboratory in which I spent so much of my activity, an attempt was made to get a lathe, a poor little amateur type of lathe, the attempt was met with an official objection that all the turning that had to be done could be done outside! However, Dr. Guthrie had his way in the end, and that poor, little, despised lathe justified its existence, and I am glad to hear that it is still in use. The change in the official attitude between that time and the present at least as to the requirements of a physical laboratory, may be seen by an examination of the splendid workshop which now exists.

The want of knowledge of the elements of science and hence of proportion and common-sense among those who have to administrate, whether in the higher official ranks in Whitehall, or in more humble local affairs, leads from time to time to fatuous acts or wanton and wicked waste. What better example can we have at a time when fuel is scarce and public waste is more than ever a scandal, than the method of darkening our streets, which after three years still prevails. No doubt there is some reduction in the light actually developed in the street lanterns, but how much reduction there may be no one can tell, for those who are responsible continue to hide their light under a bushel, and produce at least a hundred times as much as they utilise.

I should like to add another example of endless discomfort and expense, not the result of any official ignorance and indifference, but of the want of sense of the community in general, and in many cases of the neglect of the architect to stand between the builder or plumber, whose indifference is so carefully cultivated, and his miserable dupe, the householder. You will not fail to realise the annual agony of the burst water pipe, which results from the builder's habit of getting all pipes to outer walls, and, not content with that, there enclosing them inaccessibly, or to placing service pipes at insufficient depth. The burst water pipe is unknown in really cold countries where the elements of sense are allowed to prevail, and it is unnecessary here. Similarly, during our rare periods of great heat, it is impossible to per-

suade people in this country to shut the heat out during the day and open to the cool of the night.

As all my contributions to Physical Science have been experimental, and I have consistently devoted myself to the art of experiment, it is possible that some observations on this may be useful. We all have our particular aptitudes, if we have aptitude at all, and it is best for general progress that each individual should, so far as it is possible, be so engaged that his aptitude has play and his activity is in consequence the more successful. In our own science there are widely different ideas prevalent in the different branches. The borderland between Physics and Chemistry, for instance, interests a mind for which the geometrical perfection of Optics or Crystallography might offer no attraction, while it may be almost incomprehensible to those who delight in the mechanical side of physics. Similarly, there are aptitudes even more distinct leading towards abstract, theoretical or mathematical development, or, on the other hand, in the direction of experiment and invention. It is those who feel that this last is the direction that they can follow with most hope of success that in particular I am now addressing. In order to succeed as an experimentalist it is necessary to find by personal experience how as many materials as possible behave under as many conditions as possible, and this can only be done by one who will practise every mechanical art and use every tool and every instrument that he can. In the first place, it will be best to learn in the spirit of the amateur mechanic who wants to make things nearly as well as the professional, and who, therefore, merely imitates the practice of the professional mechanic, but this must not last too long or the would-be experimentalist will not cease to be the amateur mechanic. When each art and the tools and methods employed have been practised so that the operator has some confidence in his judgment as to what will happen, he may and should then use them as required, not as the amateur mechanic does imitating the professionals, and trying with probably less skill to make his work look as much like theirs as possible, but, defying all the traditions of the art, utilise the information so gained to his own ends to help him in his investigation, even though intentionally he does it in a way that would horrify the professional. It is this slavery to tradition and practice that makes the assistance of the professional so tiresome to the experimentalist. So often it takes less time to do without such assistance. I have

been constantly impressed by an expression due to Fresnel, quoted to me by a member of this Society : " If you cannot saw with a file or file with a saw, you will be no good as an experimentalist," or words to that effect. If any of our younger members should have any lurking suspicion that understanding and practice of the mechanical arts which is essential to manipulative skill is derogatory, I would urge him to consider what Fresnel did for the science of Optics, and to take his advice to heart. Again, I would urge him to think of what Newton did for Mathematical and Physical Science, and yet such was his experimental aptitude that he made his reflecting telescope (now a cherished possession of the Royal Society) with his own hands, and I believe discovered the use of that most unexpected but necessary substance pitch,* as the material for the polishing tool. Let him think of the helplessness of an experimental chemist who could not perform the elementary operations in glass-blowing, or of the experimental genius of Cavendish or Faraday if he thinks devotion to the art of experiment unnecessary. I know the very natural feeling of distrust in his own ability to construct or to extemporise devices that will really work at all. Even though he may never attain the constructive skill of the professional in any one art, he must practice as required every art, and he should arrive at results as he wants them which meet his requirements better than anything he could get done, because he knows exactly what he wants, and far more quickly as he will, when he has got over his amateur mechanic days, refuse to waste time on conventions of finish or of unnecessary elaboration. Let him beware of listening to the mischievous advice of another whose fingers perhaps are all thumbs, not to waste his precious and valuable time in doing the work of a mechanic, for the experimental art for which he is fitting himself is not the art of a mechanic, but it is the constant use of the knowledge, skill and experience of the mechanic, so far as it can be attained, guided by science and first principles, to produce new or better results than the professional can give him. I do not wish to dwell too much upon any of my own experiences in illustration of this, but I hope I may without offence refer to the development of the radiomicrometer. When I had considered the theoretical possibilities and determined the direction to follow, I found that the success of the instrument should be the greater as the thermo-

* I am informed by our member, Mr. T. Smith, of the National Physical Laboratory, that Newton probably learned of the use of pitch from Italy.

electric bars could be made smaller in cross-section. The antimony-bismuth alloys so greatly exceeded in thermo-electric power any ductile metals that I knew that I wished to use these, and at the time, having no experience in making delicate thermo-electric couples, I asked the chief designer of one of our leading instrument makers what was the smallest bar that they could make and solder. The cross-section was one or two square millimetres. Now it was essential that I should use bars of not more than about one-twentieth of a square millimetre in section, and I was assured that I could not get that done. It did not take many days to find out how to produce and solder such bars, employing not more than a fifth of a milligramme of solder for each junction, but not by imitating the professional in any respect. I remember that when I was making my early experiments in this class of construction a certain instrument maker called, and I showed him what I wanted to do. He said I had far better let him give it to his really skilled mechanics to do it properly, and not waste my valuable time in vain attempts. Naturally I paid no attention and soon found out how to succeed. And then I was brought up short by the non-existence of any suitable suspending fibre. This led to a further investigation from which the quartz fibre was evolved. First, there was the necessity for making a non-extensible and elastic fibre with a stiffness if necessary one-thousandth or one-ten-thousandth of that of spun glass. Then the trial of materials to find the best. Only glass and the siliceous minerals possessed the property of drawing out into threads, and silica alone when the difficulties of working it were overcome, was found to be the material *par excellence*. Now, I ask the superior person what chance should I have had in arriving at any of these results by going to the professional mechanic. The experimental art, the importance of which I am urging, cannot be done entirely by deputy, nor can it be understood if it is not practised. If any of our members should doubt the truth of what I am saying, he will find innumerable examples of experimental success depending on the experimental art if he will read the published volumes of Lord Rayleigh's researches, in which one of the most inspiring features is the fact that in one research after another existing knowledge is increased by home-made means, which are profoundly delicate or accurate, as the case may be, but are of surpassing simplicity.

I have made it my business to use every tool and instrument

and to handle every material that I could, and I know no other way of finding out the possibilities of construction. Though written so long ago the famous first three volumes of Holtzappfel are still the best storehouse of information on processes at large, and I take this opportunity of expressing my gratitude for all that I learned from them. I remember on one occasion having the opportunity, not by any means common, of handling five or six large uncut diamonds each as big as a walnut. Now, after getting over the surprise of noticing that diamonds are so heavy, though I knew it as a fact, the first thing to arrest attention was the curious result of bringing two into contact. Of course, everyone who has reached the early stage in manipulation of blowing glass bulbs has noticed the difference in the contact of freshly-blown glass bulbs and of bulbs which have been made any length of time. But the contact of diamonds is unlike either. There is a peculiar slipperiness, due, I think, to the feature I am now about to discuss, which is found to a less degree in a heap of flints. When brought into contact while held lightly in the fingers they emit a curious note, running up into a kind of interrogatory or falsetto squeak. Doing the same with softer materials leads to sounds with most of which we are familiar, but the diamond is unlike any. The soft touch of two corks, or the dead contact of two bullets of lead indicate naturally enough the softness or want of elasticity of these materials, and yet they are each elastic in their way. Harder materials, such as cast brass, and then gunmetal, again produce sounds that differ from the others and from one another, and the sounds give the sensation of increasing hardness or elasticity. Glass and dead-hard steel differ again, but they are not like diamond. There is some sense of bouncing, but it is not till the diamond is used that this squeak is heard. My observation was made years ago, and the opportunity did not last many minutes, so I cannot do more now than give a general impression, though no doubt what was a new experience to me is commonplace enough among those who habitually handle such property. My impression is that the note produced must have run up, perhaps, to 2,000 vibrations per second. Just as a ball under gravity bouncing from the floor makes smaller and smaller excursions at diminishing intervals proportional to the square root of the height, so each diamond lightly held by the soft fingers is under a fairly uniform force directing it towards the other one. I do not remember whether this force was

likely to be equal to or be much more than their weight, but, supposing for the moment it to be equal, see what this note means. If the note ever does reach the 2,000 vibrations per second, which is the impression left with any such steady applied force, it means that the diamonds are bouncing when so heard with slowly diminishing excursions of 1/80,000 inch, a phenomenon only possible with a material of such perfect elasticity or hardness. I have been able to turn the information obtained from this observation to account in a curious direction, to which I cannot refer, but I am now mentioning it because it is possible that some light might be thrown on the subject of the determination of hardness of hard materials by tests of this kind. It is true that all that is really observed is the approach to perfection of the coefficient of restitution, but as the superlative hard material has this quality, the question arises whether the two do not go together. Coefficient of restitution or elasticity we do understand in the sense that we can define it exactly and give it dimensions, but hardness is despairingly elusive. We all know the scale of hardness in minerals in which the diamond is pre-eminent. The harder material will scratch the softer, but not be scratched by it. Again, we know that there are certain empirical methods of testing the hardness of metals in the workshop. Of these, the Brinell test, which depends on the measurement of the depression made by a steel ball under pressure is no doubt the best known. There is another test with an instrument called the Shore Scleroscope, which depends on the rebound of a diamond-faced hammer. Another class of test depends on a scratching process, and yet another on resistance to wear. But the results of different tests do not necessarily agree, because though intended to measure hardness they are in reality measuring something different. The subject is discussed very fully in the " Proceedings " of the Mechanical Engineers for November, 1916, a copy of which was kindly sent to me by Sir Robert Hadfield.

The general conclusion come to is that tenacity and elasticity are main factors determining the result of a test. It is also found that when the hardness of dead-hard tool steel is reached the methods of testing have come to their limit. The subject is so curiously involved that it may be well to consider the scratching test. A corner of quartz will scratch hard steel and a sandstone grinder will grind steel. On the other hand, if the scale of pressure is altered the steel tool will

plough out a groove in the sandstone. Here no doubt the structure of the hard grains and cementing material is less strong than the material of the grains, and this may be a true explanation so far, but I believe that a properly held steel tool very forcibly applied to a well-supported large crystal of quartz would crush the harder material in front of it, and so could be made to scratch the quartz crystal. If so, what is the meaning of any scratch test ?

At any rate, there is a well-known tool used largely for turning granite and hard rock, in which a hard steel disc dished and narrow at the edge rolls under pressure over the material, which comes away in flakes, while the steel is undamaged. Another factor of the utmost importance is speed, but it would take me too far away to pursue the discussion to this. It is difficult to define hardness. It comes back to a question of tenacity, resistance to crushing or to shearing. This is true of the Brinell test, while the Shore test introduces also elasticity, but it is not only elasticity, for it requires that a permanent indentation be made and work done on the material leading to a reduced rebound. The question arises whether there is any such thing as hardness at all—*i.e.*, in the physical sense—exactly definable and having dimensions. I am almost forced to think not, and that what is appreciated as hardness is a variable and unknown combination of these other qualities. Hardness is the Mrs. Harris of the workshop. This does not prevent the workshop determination of hardness of metals from being of great practical importance, and it is the more complicated by the crystalline structure of metals. For this reason Sir Robert Hadfield has offered a prize for a new and better workshop test of hardness of metal, more especially for the harder kinds, to which I am glad to direct your attention. Now, going back to the diamond and its squeak, it seems possible that a determination under defined conditions of the musical rebound would enable one to differentiate between the hardness of very hard materials from the diamond down to steel or even lower. Where conducting bodies are concerned the use of a Duddell oscillograph would enable one to determine the period of rebound and the rapidity of its diminution, and hence under exactly known pressure conditions the coefficient of restitution. According to the hardness there would be, I expect, an abrupt stoppage of the actual bouncings, so that if for instance, the last bounce of diamond was found to be 1/100,000 inch, sapphire, topaz, quartz and steel would in order

show more rapid diminution of period and a larger final excursion. Of course, any such test is unlike tests which depend upon permanent deformation, and being entirely within the elastic limit, it is possibly useless as a workshop test for metals, but it might give some useful information on hard minerals and crystals. In the case of non-conducting crystals the only way of connection with the oscillograph would be through a microphone, for the diminishing period only need be determined; the diminishing and final terminal amplitude then follow.

Hardness is not the only elusive physical quality which is perplexing engineers at the present time. When the worm-gear designed by Lanchester was tested for efficiency at the National Physical Laboratory in that beautiful testing machine invented and designed by Lanchester also, a machine which directly measures the loss, instead of leaving it to be found as the small difference between two imperfectly known large quantities, it was found that the lubricating property of oil depended on something which at present is unknown. It is not viscosity. Animal and vegetable oils lubricate better—*i.e.*, they are more slippery than mineral oils of the same viscosity—and though the oil trade has known how to make good slippery mixtures no one at present knows what "oiliness" is, and this is at the present time an important physical quest of the engineer. It may, for instance, be as real as hardness, in being a combination quality, and yet share with hardness its relationship with Mrs. Harris.

Here, then, are two very difficult and at present elusive subjects of inquiry, for which the engineer seeks a practical answer, but in so far as any practical answer is based upon exact principle and strict definition will it be likely to approach to a true and complete answer, and this part of the answer is one which Members of this Society might seek for, for it is clearly within the domain of physical science.

While there is always abundant scope for the better utilisation of scientific principles in our manufactures, I would most earnestly caution enthusiasts from our science schools against supposing that manufacturers are ignorant, and that they can easily put them right. I have probably seen more of the development of invention in manufacturing operations in the last quarter of a century than the majority of those concerned in the teaching of science, and I am bound to say that I have been struck on the whole not by the ignorance prevailing, but by the wonderful ingenuity and thorough understanding of what

they are doing to be met with in the great factories. Not only, as the scale is increased and special knowledge is impressed, do we find the scientific equipment in men and things the more perfect, but when fiscal conditions allow and the scale is sufficient, the tables are turned, and discovery and research of value to science are made in the factory which would be out of the reach of our scientific institutions or possibly even of our national laboratories, simply on the ground of expense and equipment. The researches connected with the name of Dr. Coolidge, at Schenectady, on tungsten, and the production of targets and the drawing of this refractory metal through red-hot diamonds, appear to me to be an example of such retaliation. The enormous cost of such researches is only possible in the development of a great industrial process, and in such cases pure science is benefited by industry.

In advocating more attention to the cultivation of experimental art, you will understand that I am not suggesting in the slightest degree that theoretical and mathematical study should be neglected. My theme is that experimental art is essential to success, and that it tends to be too much neglected. Just as the classic schoolmaster takes charge in our schools, and makes every other subject give way to his favourite classics, so there is danger of our more theoretical professors failing to realise that there is an immense lore on which the experimental art is based, and that successful experiment is an essential element of scientific progress.

VII. *Recording Thermometer.* *By* C. V. Boys, *F.R.S.*

Received Nov. 15, 1917.

The instrument described in this Paper was designed to meet a special case, and while there is no new scientific principle involved it is so simple and so easily made and its operation is so good that a description of it may be of some interest. The object sought was to obtain a continuous record of the temperature in the case of a regulator clock which I knew to be over-compensated for temperature, but I wished to have something better than readings of a thermometer in the clock at week-ends only, so as to compare the rate as determined by a transit instrument with the mean temperature. I have a fairly good idea of the amount of over-compensation from about 18 months' comparison of rate with the weekly temperature readings, but a continuous temperature record would be much more satisfactory.

Inside a clock-case the pendulum and falling weight break up the somewhat restricted space available, so that an instrument must be adapted to the situation; further it is desirable that the thermometer part of the instrument should extend over about the same limits of level as the gridiron and bob of the pendulum rod. I have, therefore, taken an ebonite rod 35 inches long and $\frac{3}{8}$ inch in diameter and threaded it through a stout glass tube. The rod is sawn through at each end axially for a short distance and cross-drilled so as to take at the lower end a bridge made of a piece of hacksaw blade and at the upper end the short arm of a lever of which the fulcrum is also a piece of hacksaw blade. A notch is filed diametrically across the glass tube at its lower end for the bridge to bear against, while a corresponding notch is filed across the glass tube at the other end about tangent to its inner surface. A brass ring is cemented to the outside of the tube a few inches from its upper end, and a spring (Fig. 2) connecting the long arm of the lever with this ring keeps both knife edges in firm contact with the file notches. This part then is a self-contained whole and the end of the long arm moves about 7 times as much as the ebonite within the tube. The tube is suspended in the corner of the clock out of reach of the pendulum from a bracket, which carries also a cantilever support adjustable in all directions for the axis of the recording pen. This also is provided with a short arm which comes below the end of

the long arm of the thermometer, and a long arm reaching across the clock between the pendulum gridiron and the weight to the pen which makes its record on a drum on the other side. The weight of the long arm of the pen causes the short arm to rise as far as it can, but it is controlled by a steel rod pointed at each end. The points rest in centre punch dots above the short arm of the pen lever and below the long arm of the thermometer lever. There are a number of these dots and it is a small matter to shift the rod from one to another until the desired scale of magnification is reached. A piece of cotton is tied to the steel rod and to the cantilever, so that if the rod is dropped it does not descend into the depths. I have found the dimensions stated below give a movement of 1 inch for 10°F. at temperatures of 40°–55° F. Having in the country no means other than the seasons for regulating the temperature, I have not at present determined to what extent the scale is more open at higher temperatures. Ultimately I should rule paper to agree with the instrument rather than try by inclination or slight bell-cranking of the levers to cause them to introduce an approximate correction. I have made the pen of silver, somewhat like a drawing pen of a box of instruments but without a screw and very small and well finished, and it rules a very fine clean line. The record can be read to $\frac{1}{20}$° F., and the whole arrangement is so rigid that it could certainly be used with more magnification to record to $\frac{1}{100}$° C. on an empirical scale, which would have to be determined.

It seemed absurd to put a driving clock into a drum already situated in a clock, and so I have made the following arrangement for moving the drum. The drum is a free fit upon a wheel with tubular centre which turns on an upright steel rod. The drum may thus be lifted off to change the paper. The wheel rests upon a washer and boss. Under the horizontal wheel a radial axis is placed with its inner pivot close to the boss and its outer end resting by means of a groove in an upturned *V* on an elastic arm. Beyond the groove and overhanging a 4-inch pulley of aluminium is fixed to the radial axis. Between the groove and the inner end a roller of ebonite about 0·3 inch in diameter is also fixed, and this is pressed by the elasticity of the arm and *V* up against the lower surface of the wheel. The two surfaces are smooth and conical, but the ebonite cone is slightly convex so that it may be adjusted and still give approximately correct rolling. A cord from the bottom of

the clock weight passes under a pulley on a weight let down so as to rest on the floor of the clock under the driving weight up over the aluminium pulley to a peculiar weight to be described later. As the clock-weight falls $1\frac{1}{2}$ inches a day the drum rotates $\frac{1}{8}$ inch a day, and so the ordinary barometer charts are available, the two-hour periods becoming days and the horizontal $\frac{1}{16}$ inch spaces being $\frac{1}{2}°$ F. In the place of 28, 29, 30, 31 inches are written 40°, 50°, 60°, 70°, but when the winter had fairly started these were made 30°, 40°, 50°, 60°, and in the summer 50°, 60°, 70°, 80° may be chosen. There is a screw adjustment under the pen carrier, shown in Fig. 2 so that it may be adjusted in a moment so as to read correctly.

I have said that the small weight is peculiar. It is desirable when the clock is wound, which need be only once a month, that the drum should not unwind and require resetting, and it is desirable that the small weight with its smaller fall should not limit the time that the clock may be left unattended. Further it is desirable that the pull on the cord below the weight should not appreciably alter the driving force of the weight. These three ends are attained by taking the cord round a pulley on the small weight and up to a nail which halves the fall, and then by fixing two pieces of wine cork very nearly in contact above the pulley and leading the cord round these two quarter turns, so that it enters and leaves adjacent to itself. Owing to the friction of nearly half a turn of cork the entering cord and the leaving cord are very differently loaded, the ratio $e^{\mu\theta}$ being perhaps 1 to 5. Taking this as an example the cord round the pulley carries only one-sixth of the small weight when the clock is going, and this is all that is added to the pull of the clock weight, while the nail takes five-sixths. The aluminium pulley is therefore pulled down as by $\frac{1}{3}$ of the small weight and the spring V support overcomes this and keeps the ebonite roller in driving contact with the drum. When, however, the clock is wound, the cord over the aluminium pulley, which had been the entering cord, becomes the leaving cord and its tension is increased five fold, and the aluminium pulley now carrying $1\frac{2}{3}$ of the small weight overcomes the spring V support and the ebonite roller moves down out of contact with the wheel carrying the drum, through about a thousandth of an inch, where it is limited by a screw stop. A second screw is provided to boost the spring if necessary (Fig. 3). It would not do simply

to pass the cord round a pulley on the small weight secured thereto so that it could not turn, because with the two cords so far apart a considerable movement would occur before the transfer of the load by the rocking of the weight was sufficient to put the spring *V* into action.

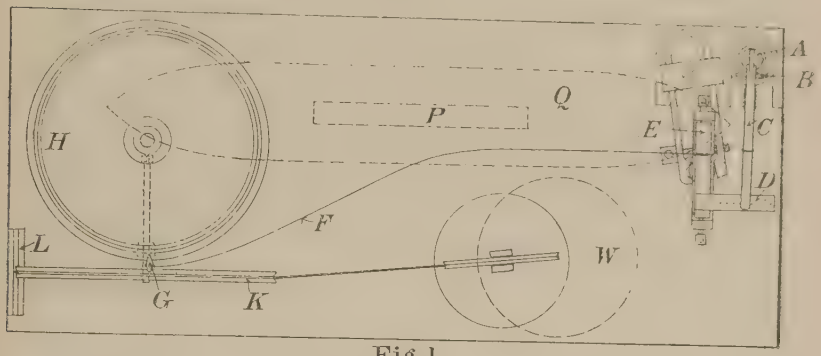

Fig.1

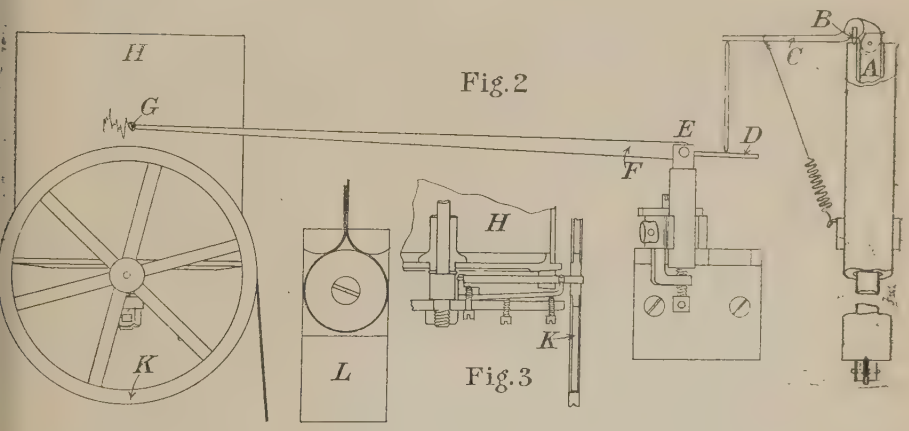

Fig.2

Fig.3

Fig.4

A, the ebonite rod in its glass tube.
B, upper knife edge.
C, long arm of thermometer lever, 2 inches (short arm about 0·3 inch).
D, short arm of pen lever 0·7 inch.
E, axis of pen lever.
F, long arm of pen lever 8½ inches.
G, pen.
H, recording drum.
K, aluminium wheel.
L, small weight.
P, space occupied by pendulum rod.
Q, space at lower level occupied by pendulum bob.
W, space occupied by driving weight.

Fig. 1 is a plan showing the position of the parts of the instrument as placed in the clock. Fig. 2 is an elevation in which the thermometer is removed from its bracket and turned round about the pointed steel rod as an axis so as to show more clearly the operation. The insets Figs. 3 and 4 show the small weight and the spring V, &c., used for driving the drum.

I should mention that ebonite was used by Edison about 40 years ago in the instrument he called a tasimeter. It has a very large coefficient of expansion which makes it a convenient material for use in instruments of this class. It would also be an extremely convenient expanding element in a compensation pendulum, but it might not be suitable for clocks of high precision on account of a slight want of proportionality in its expansion.

I show in Fig. 5 a design for a temperature recorder to go

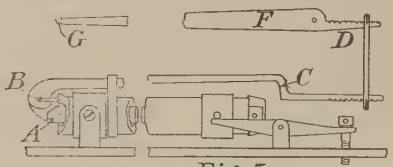

Fig. 5

into an ordinary barograph in front of the barograph mechanism. It seems hardly necessary to give any further description. A convenient scale to aim at would be $1^{\circ}\text{F.} = \frac{1}{20}$ inch, so that the 3 inches would cover 60°. The pen would be $\frac{1}{8}$ inch in advance of the barograph pen, so that the two records on the same paper would differ in time by two hours and different coloured inks might be used. The connecting bar being in this case in tension would hook over notches as shown, and the right notches would be found by trial. The glass tube is shown cemented into a collar near one end resting on trunnion pivots, and it can be raised or lowered by a screw lever at the other end to adjust the position of the pen on the scale. A spring band to the right of the collar draws the long arm of the thermometer lever towards the glass tube, thus keeping the two knife edges in firm contact with their V's. With a length of ebonite of 6 inches a magnification of 220 would be needed, which would conveniently be obtained with a first lever magnification of 20 and a pen lever magnification of 11.

It will be noticed that with this design the magnification increases with the square of the length.

ABSTRACT.

This instrument was designed and constructed to go into the case of a regulator clock. The thermometric element consists of a rod of ebonite within a glass tube. The differential expansion is determined by a pair of levers giving a movement of 1 in. for 10° F. The drum carries an ordinary barometer chart, and is driven at such a speed that a two-hour interval of $\frac{1}{8}$ in. is passed in 24 hours. The drum is driven by friction by means of a cord from below the driving weight of the clock by an $e^{\mu\theta}$ arrangement, in virtue of which when the clockweight descends the drum turns, but when the clock is wound the drum remains at rest. The instrument is designed with a view to easy construction and accuracy. It is extremely rigid, and much more magnification might be used.

An alternative design on the same lines to go into a recording barograph is also given.

DISCUSSION.

Mr. C. C. PATERSON said that for certain purposes a quick-acting sensitive recording thermometer was very useful. What was the time lag of this instrument in taking up the temperature of the surroundings ?

Prof. LEES asked how the ebonite behaved as regards constancy.

Prof. BOYS, in reply, said that for his purpose the thermometer was desired to be slow and sluggish, so as not to take up every trifling variation of temperature due to people entering the room. To make it quick acting, a thin strip of ebonite would have to be used, and the glass tube should be replaced by a stout glass rod or plate some distance from the ebonite. As regards constancy, he could only speak for the three months that the instrument had been in use. There had been no signs of variability in that time.

VIII. *The Primary Monochromatic Aberrations of a Centred Optical System.* *By* S. D. Chalmers, *M.A., Technical Optics Department of the Northampton Institute.*

Received November 14, 1917.

In considering the defects of an optical system the first step is to obtain a simple theory of optical systems, ignoring all those effects which tend to produce bad definition or distortion of the image. This theory should furnish the ideal positions and sizes of the images to serve as comparisons with the actual results obtained by the system.

The next step is to take into account the most important of the causes which contribute to the defects of simple lenses when they are used with moderate apertures and fields. These defects will be referred to as Aberrations of the First Order.

In practical systems these defects are usually compensated by other lenses, and the residual defects are made up of the

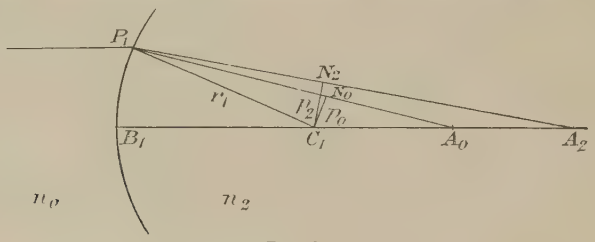

Fig. 1.

uncompensated effects of the First Order and other defects which are ignored in this treatment, and will be referred to as Aberrations of Higher Order.

In the treatment of the First Order Aberrations two methods will be used, one primarily suited to the case where the separation of the surfaces is small, and the other more suitable to the case where the separations vitally affect the design.

The aberrational defects are expressed as lateral aberrations, *i.e.*, as defects measured in the focal plane of the system. The procedure adopted is to express the aberrational defect of a single surface in terms of the constants of the surface and the perpendicular distance of the ray considered from the centre of curvature of the surface.

The value of this perpendicular can be expressed in terms of the co-ordinates of the ray in any chosen medium, and thus

the aberration due to each surface can be expressed in terms of the co-ordinates of the chosen ray, in such a way that the aberrations of the individual surfaces can be summed.

Refraction at a Single Surface.

Let P_1B_1 (Fig. 1) be a spherical surface of radius r_1 separating the two media n_0 and n_2. Let C_1 be the centre of curvature, B_1 the vertex and P_1 any point on the surface, and let the ray which was travelling in the direction P_1A_0 in medium n_0 be deviated to P_1A_2 in medium n_2.

Denoting the angles $C_1P_1A_0$ by i_1 and $C_1P_1A_2$ by j_1 the law of refraction gives

$$n_0 \sin i_1 = n_2 \sin j_1.$$

In the triangle $C_1P_1A_0$, we have—

$$\frac{\sin C_1}{\sin i_1} = \frac{P_1A_0}{C_1A_0},$$

so that—

$$\frac{P_1A_0}{n_0C_1A_0} = \frac{\sin C_1}{n_0 \sin i_1} = \frac{\sin C_1}{n_2 \sin j_1} = \frac{P_1A_2}{n_2C_1A_2}. \quad . \quad . \quad (1)$$

If the perpendiculars $C_1N_0 \equiv p_0$ and $C_1N_2 \equiv p_2$ be dropped from C_1 on P_1A_0 and P_1A_2 respectively,

$$P_1A_0 = (C_1P^2 - p_0^2)^{\frac{1}{2}} + (C_1A_0^2 - p_0^2)^{\frac{1}{2}},$$

and this expression can be expanded in powers of p_0^2,

$$P_1A_0 = C_1P + C_1A_0 - \tfrac{1}{2}p_0^2\left(\frac{1}{C_1P_1} + \frac{1}{C_1A_0}\right) + \text{terms in } p_0^4, \text{ \&c.}$$

Similarly,

$$P_1A_2 = C_1P + C_1A_2 - \tfrac{1}{2}p_2^2\left(\frac{1}{C_1P_1} + \frac{1}{C_1A_2}\right) + \ . \ . \ . p_2^4, \text{ \&c.}$$

Writing $C_1P = r_1$ we have from equation (1)

$$\frac{r_1 + C_1A_0}{n_0C_1A_0} - \tfrac{1}{2}\frac{p_0^2}{n_0C_1A_0}\left(\frac{1}{r_1} + \frac{1}{C_1A_0}\right) + \ . \ . \ .$$

$$= \frac{r_1 + C_1A_2}{n_2C_1A_2} - \tfrac{1}{2}\frac{p_2^2}{n_2C_1A_2}\left(\frac{1}{r_1} + \frac{1}{C_1A_2}\right) + \ . \ . \ . \ . \ . \quad (2)$$

and we shall approximate at first by ignoring all terms in p_0^2 and p_2^2, but shall take these terms into account later on. If the terms in p_0^2 and p_2^2 can be ignored, all rays proceeding to A_0 will be deviated to A_2, whatever be the position of P_1, and A_2 will be the image of A_0.

To this approximation

$$\frac{r_1+C_1A_0}{n_0C_1A_0}=\frac{r_1+C_1A_2}{n_2C_1A_2}. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Dividing by r_1 this can be expressed as

$$\frac{1}{n_0C_1A_0}+\frac{1}{n_0r_1}=\frac{1}{n_2C_1A_2}+\frac{1}{n_2r_1}. \quad . \quad . \quad . \quad (4)$$

If a small object A_0E_0 at right angles to the axis be considered, and a plane be drawn through A_2 at right angles to the axis, a ray proceeding to E_0 and passing through C_0 will not be deviated ; it will therefore meet the plane through A_2 in E_2, such that

$$\frac{A_2E_2}{A_0E_0}=\frac{C_1A_2}{C_1A_0}.$$

The point E_2 will be a point in the image patch of E_0, and may be regarded as the ideal image of E_0. In this case the object A_0E_0 is reproduced of the size A_2E_2, and $\frac{A_2E_2}{A_0E_0}$ is termed the linear magnification, and will be denoted by m_1.

For convenience we use the value $\frac{n_2A_2E_2}{n_0A_0E_0}$, and denote this by M_1.

The quantities $\frac{1}{n_0C_1A_0}$ and $\frac{1}{n_2C_1A_2}$ are denoted by S_1 and T_1 respectively, and the relation (4) becomes $T_1-S_1=\left(\frac{1}{n_0}-\frac{1}{n_2}\right)\frac{1}{r_1}$, and the last term being independent of A_2 and A_0 is a property of the surface only, and may be denoted by P_1.

We thus have $$T_1-S_1=P_1$$

and $$M_1=S_1/T_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From these two relations we deduce

$$M_1=1-\frac{P_1}{T_1},$$

$$\frac{1}{M_1}=1+\frac{P_1}{S_1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or, writing $t_1=\frac{1}{T_1}$ and $s_1=\frac{1}{S_1}$

$$M_1=1-t_1P_1 \quad \frac{1}{M_1}=1+s_1P_1.$$

These formulæ can be applied to any system of lenses; let $P_{1\cdot3}=T_1/M_3$, $n_2C'_1C'_3=a_2$, &c., and let $M_3s_3t_3P_3S_3$ and T_3 relate to the next surface.

Then, since $\dfrac{1}{M_3}=1+\dfrac{P_3}{S_3}$,

$$T_{1\cdot3}=(1+s_3P_3)T_1=T_1+\frac{t_1}{t_1}\frac{a_2}{}P_3.$$

If a ray be drawn through the system from the object point on the axis, so that the perpendicular distance from C'_1 is $-p_0$ then the optical length of this perpendicular would be n_0p_0, and this would equal the corresponding quantity after refraction; denoting this quantity by q_1 and the corresponding quantities for the other surfaces by q_3, we have

$$\frac{q_3}{q_1}=\frac{s_3}{t_1}=1+a_2T_1,$$

so that the quantity $1+a_2T_1$ represents the optical length of the perpendicular from C_3, as compared with the optical length of the perpendicular from C_1.

The method can be extended to more surfaces, and we have

$$T=T_1+\frac{q_3}{q_1}P_3+\frac{q_5}{q_1}P_5, \quad \ldots\ldots\ (7)$$

and

$$\frac{q_{2r-1}}{q_1}=\frac{q_{2r-3}}{q_1}=a_{2r-2}T_{1,2r-3}. \quad \ldots\ldots\ (8)$$

Also

$$t_{2r-1}=\frac{q_{2r-1}}{q_1}/T. \quad \ldots\ldots\ *(9)$$

If $S_1=O$, $T_1=P_1$, and T is analogous to the equivalent focal power of the system, denoting this value of T by P, we have

$$M=\frac{q_{2r-1}}{q_1}-t_{2r}\ P, \quad \ldots\ldots\ (10a)$$

as the relation between M, P and T_{2r-1} for the complete system, where $\dfrac{q_{2r-1}}{q_1}$ and P are calculated for $S_1=o$, but M and T_{2r-1} relate to any object point.

* NOTE.—p_1 would then relate to any ray, while q_1 relates only to this selected ray drawn from an object point on the axis.

The analogous relation between s_1 and M is obtained by considering the reversed system, and is

$$\frac{1}{M}=\frac{Q_1}{Q}+s_1P, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10b)$$

where Q_1 and Q denote the optical lengths of the perpendiculars from C_1, and the centre of curvature of the last surface, on a ray incident parallel to the axis in the last medium.

Aberrations of a Single Surface.

Returning to equation (2) we have to the next approximation

$$\frac{r_1+C_1A_2}{n_2C_1A_2}=\frac{r_1+C_1A_0}{n_0C_1A_0}+\tfrac{1}{2}\frac{p_2^2}{n_2C_1A_2}\left(\frac{1}{r_1}+\frac{1}{C_1A_2}\right)-\tfrac{1}{2}\frac{p_0^2}{n_0C_1A_0}\left(\frac{1}{r_1}+\frac{1}{C_1A_0}\right),$$

on dividing by r_1

$$\frac{1}{n_2C_1A_2}+\frac{1}{n_2r_1}-\left(\frac{1}{n_0C_1A_0}+\frac{1}{n_0r_1}\right)=+\tfrac{1}{2}\frac{p_2^2}{n_2\,.\,C_1A_2\,.\,r_1}\left(\frac{1}{r_1}+\frac{1}{C_1A_2}\right)$$
$$-\tfrac{1}{2}\frac{p_0^2}{n_0\,.\,C_1A_0\,.\,r_1}\left(\frac{1}{r_1}+\frac{1}{C_1A_0}\right), \quad (11)$$

but $n_2p_2=n_0p_0$, and this may be denoted by p_1.

Also $$\frac{1}{n_2C_1A_2}+\frac{1}{n_2r_1}=\frac{1}{n_0C_1A_0}+\frac{1}{n_0r_1}$$

to the first approximation.

Substituting this in the terms of higher order the value of $\frac{1}{n_2C_1A_2}$ exceeds its ideal value by

$$\delta\frac{1}{n_2C_1A_2}=+\tfrac{1}{2}\frac{p_1^2}{r_1}\cdot\left(\frac{1}{n_2^2C_1A_2}-\frac{1}{n_0^2}\,\frac{1}{C_1A_0}\right)\left(\frac{1}{r_1}+\frac{1}{C_1A_2}\right)\frac{1}{n_2}.$$

And

$$\delta\frac{1}{n_2C_1A_2}=-\frac{1}{n_2C_1A_2^2}\cdot\delta C_1A_2$$
$$=\tfrac{1}{2}\frac{p_1^2}{r_1}\left(\frac{1}{n_2^2C_1A_1}-\frac{1}{n_0^2}\,\frac{1}{C_1A_0}\right)\left(\frac{1}{r_1}+\frac{1}{C_1A_2}\right)\frac{1}{n_2}.$$

But the lateral aberration is parallel to p_2 and $=\frac{p_2}{C_1A_2}\cdot\delta C_1A_2$,

so that the lateral aberration is

$$\frac{p_1}{n_2 C_1 A_2} \times -n_2 C_1 A_2{}^2 . \frac{1}{2} \frac{p_1{}^2}{r_1} \left(\frac{1}{n_2{}^2 C_1 A_2} - \frac{1}{n_0{}^2} \frac{1}{C_1 A_0} \right) \left(\frac{1}{n_2 r_1} + \frac{1}{n_2 C_1 A_2} \right)$$

$$= p_1 \left(-\tfrac{1}{2} p_1{}^2 \frac{C_1 A_2}{r_1} \right) \left(\frac{1}{n_2{}^2 C_1 A_2} - \frac{1}{n_0{}^2} \frac{1}{C_1 A_0} \right) \left(\frac{1}{n_2 r_1} + \frac{1}{n_2 C_1 A_2} \right).$$

This is measured parallel to p_2 and its co-ordinates are obtained by substituting the co-ordinates of p_1 in the first term of this expression.

Thus denoting these co-ordinates of the aberration by $[\Delta]$ we have

$$[\Delta] = -\tfrac{1}{2} [p_1] \frac{C_1 A_2}{r_1} p_1{}^2 \left(\frac{1}{n_2{}^2 C_1 A_2} - \frac{1}{n_0{}^2} \frac{1}{C_1 A_0} \right) \left(\frac{1}{n_2 r_1} + \frac{1}{n_2 C_1 A_2} \right) \tag{12}$$

Rays from a Point Not on the Axis.

This expression will also give the aberration of an oblique ray, relative to the ray which passes through the centre of

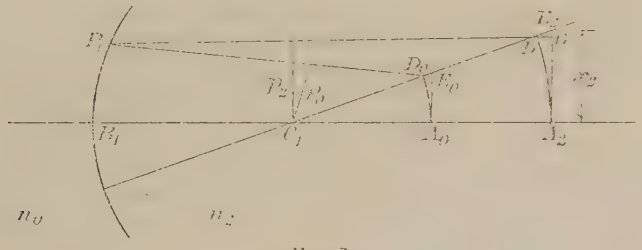

Fig. 2.

curvature, provided this aberration be measured at the sphere of radius $C_1 A_2$. But it is usually more convenient to express this difference on the focal plane, and there arise additional terms which must be included.

If we represent the distances of the image and object from the axis by a_0 and x_2 we have from Fig. 2.

$$EG = \frac{-p_2}{C_1 A_2} \times ED = \frac{-p_2}{C_1 A_2} \times \frac{x_2{}^2}{2 C_1 A_2},$$

but there is a similar term

$$\frac{-p_0}{C_1 A_0} \quad \frac{x_0{}^2}{2 C_1 A_0},$$

which affects the ray before refraction, and this must be multiplied by $\frac{x_2}{x_0}$, the magnification produced by the surface, and $\frac{x_2}{x_0}=\frac{C_1A_2}{C_1A_0}$ to the first approximation.

Thus, the additional term is

$$\frac{x_2^2}{2C_1A_2}\left\{\frac{-p_2n_2}{C_1A_2n_2}+\frac{p_0n_0}{n_0C_1A_0}\right\}=\frac{-p_2n_2}{2C_1A_2}x_2^2\left\{\frac{1}{n_2C_1A_2}-\frac{1}{n_0C_1A_0}\right\},\quad(13)$$

and once more the co-ordinates of p_2 can be substituted for p_2 to give the co-ordinates of the lateral aberration. Thus, the complete lateral aberration arising from any surface—say, (5)—and measured on the image plane in medium (6) is

$$[\Delta_6]=-\tfrac{1}{2}[p_5]\left\{\frac{C_5A_6}{r_5}(p_5)^2\left(\frac{1}{n_6^2C_5A_6}-\frac{1}{n_4^2C_5A_4}\right)\left(\frac{1}{n_6C_5A_6}+\frac{1}{n_6r_5}\right)\right.$$
$$\left.+\frac{x_6^2}{C_5A_6}\left(\frac{1}{n_6C_5A_6}-\frac{1}{n_4C_5A_4}\right)\right\}.\quad(14)$$

This can be expressed in terms of s and t as follows :—

$$[\Delta_6']\equiv[n_6\Delta_6]=-\tfrac{1}{2}[p_5]\left\{\frac{t_5}{r_5}p_5^2\left(\frac{T_5}{n_6}-\frac{S_5}{n_4}\right)\left(T_5+\frac{R_5}{n_6}\right)\right.$$
$$\left.+(n_6x_6)^2T_5(T_5-S_5)\right\},\quad\ldots\quad(15)$$

and

$$T_5-\frac{R_5}{n_6}-S_5+\frac{R_5}{n_4}=T_5+\frac{1}{n_6}\frac{T_5-S_5}{\frac{1}{n_4}-\frac{1}{n_6}}=\frac{T_5\frac{1}{n_4}-S_5\frac{1}{n_6}}{\frac{1}{n_4}-\frac{1}{n_6}}$$
$$=\frac{n_6T_5-n_4S_5}{n_6-n_4},$$

also $$T_5-S_5=P_5.$$

If we desire to express $[\Delta_6']$ in the last medium we must multiply by each subsequent M, *i.e.*, by $T_{1\cdot3\cdot5}/T$, or n_6x_6 by $\frac{nx}{n_6x_6}$.

We have $$T_5=T_{1\cdot3\cdot5}\frac{q_1}{q_5}\quad S_5=T_{1\cdot3}\frac{q_1}{q_5},$$

so that $[\Delta']_6$ the co-ordinates in the last medium of the aberration due to surface 5 are

$$[\Delta']_6 = -\tfrac{1}{2}\left[\frac{p_5 q_1}{q_5}\right]\left\{\left(\frac{p_5 q_1}{q_5}\right)^2 \frac{1}{r_5}\left(\frac{T_{1\cdot3\cdot5}}{n_6} - \frac{T_{1\cdot3}}{n_4}\right)\left(\frac{n_6 T_{1\cdot3\cdot5} - n_4 T_{1\cdot3}}{n_6 - n_4}\right)\right.$$
$$\left. \frac{1}{T}\left(\frac{q_5}{q_1}\right)^2 + (nx)^2 T_5\right\}. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The coefficient

$$\frac{1}{r_5}\left(\frac{T_{1\cdot3\cdot5}}{n_6} - \frac{T_{1\cdot3}}{n_4}\right)\left(\frac{T_{1\cdot3\cdot5} n_6}{n_6 - n_4} - \frac{T_{1\cdot3} n_4}{n_6 - n_4}\right)\frac{1}{T}$$

is in a form suitable for calculation, but it is sometimes more convenient to express it as

$$\frac{R_5 \cdot n_6 - n_4}{(n_6 - n_4)^2}\left(T_{1\cdot3\cdot5}{}^2 - T_{1\cdot3} T_{1\cdot3\cdot5}\left(\frac{n_6}{n_4} + \frac{n_4}{n_6}\right) + T_{1\cdot3}{}^2\right)\frac{1}{T}$$
$$= \frac{1}{T}\,\frac{P_5 \cdot n_4 n_6}{(n_6 - n_4)^2}\left\{(T_{1\cdot3\cdot5} - T_{1\cdot3})^2 - T_{1\cdot3} T_{1\cdot3\cdot5}\,\frac{n_4{}^2 - 2n_4 n_6 + n_6{}^2}{n_4 n_6}\right\}$$
$$= \frac{1}{T}\,\frac{P_5 \cdot n_4 n_6}{(n_6 - n_4)^2}\left\{\left(\frac{q_5 P_5}{q_1}\right)^2 - T_{1\cdot3} T_{1\cdot3\cdot5}\,\frac{(n_4 - n_6)^2}{n_4 n_6}\right\}$$
$$= \frac{1}{T}\;P_5\left\{\left(\frac{q_5 P_5}{q_1}\right)^2 \frac{n_4 n_6}{(n_6 - n_4)^2} - T_{1\cdot3} T_{1\cdot3\cdot5}\right\}. \quad . \quad . \quad . \quad . \quad (17)$$

It now remains to express $p_5 \dfrac{q_1}{q_5}$ in terms of the quantities which can be readily calculated for an individual ray.

Using optical lengths throughout, let x_0 be any object point, $x_1 y_1$ the co-ordinates at which it passes the plane (1).

Let p_5' be the image of p_5 in the medium O and let its distance from the plane (1) be b_5, then, from Fig. 3 the co-ordinates of p_5' are

$$[p_5'] = x_0 \frac{b_5}{s} + x_1 \frac{s - b_5}{s},\; y_1 \frac{s - b_5}{s},$$

where s is the distance from plane (1) to the object.

If the intercepts of the ray from which q_1, q_3, q_5 are calculated on the image planes of 1, 3, 5, &c., in medium O be, we have

$$\frac{p_5 q_1}{q_5} = \frac{p_5' q_1}{q_5} \quad \text{and} \quad \frac{q_1}{q_5'} = \frac{s}{s - b_5},$$

so that—

$$\left[\frac{p_5 q_1}{q_5}\right] = x_1 \frac{s - b_5}{s}\,\frac{q_1}{q_5'} + x_0 \frac{b_5}{q_5'}\,\frac{q_1}{s},\; y_1 \frac{s - b_5}{s}\,\frac{q_1}{q_5'} = x_1 + \frac{x}{s}\,\frac{b_5 q_1}{q_5'}, y_1. \quad (18)$$

If we denote the co-ordinates x_1y_1 by $\rho\cos\psi$ and $\rho\sin\psi$ and let the ray from $x_0=O$, through x_1y_1 make the angle A, with the axis in the final medium, then

$$x_1T=A\cos\psi \quad \text{and} \quad y_1T=A\sin\psi,$$

where the approximations do not determine whether the sine, circular measure or tan of A is to be taken.

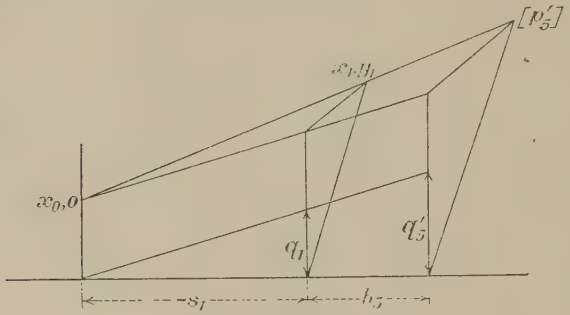

FIG. 3.

Writing $\frac{b_5q_1}{q_5'}T=B_5$, we have

$$[\Delta']_6=-\tfrac{1}{2}\frac{1}{P}[A\cos\psi+xTB_5A\sin\psi]\Big\{(A^2+2AxT\cos\psi B_5 +(xT)^2B_5^2)\Big(\Big(\frac{q_5}{q_1}\Big)^2\frac{P_5}{P}\Big(\Big(\frac{q_5}{q_1}\Big)\frac{P_5}{P}\Big)^2\frac{n_4n_6}{(n_6-n_4)^2}-\Big(\frac{T_{1\cdot3\cdot5}T_{1\cdot3}}{T^2}\Big)\Big) +\frac{P_5}{T}(xT)^2\Big\}. \quad . \quad . \quad (19)$$

Thus, writing E_5 for the expression

$$\frac{q_5}{q_1}\frac{q_5P_5}{q_1T}\Big(\Big(\frac{q_5}{q_1}\frac{P_5}{T}\Big)^2\Big(\frac{n_4n_6}{(n_6-n_4)^2}\Big)-\frac{T_{1\cdot3\cdot5}T_{1\cdot3}}{T^2}\Big),$$

we have the two co-ordinates

$$(\delta nx)_6=-\tfrac{1}{2}\frac{1}{T}\Big\{E_5A^3\cos\psi+E_5B_5A^2xT(2\cos^2\psi+1)+3E_5B_5^2A\cos\psi(xT)^2+EB_5^3(xT)^3+\frac{P_5}{T}A\cos\psi(xT)^2+B_5\frac{P_5}{T}(xT)^3\Big\}$$

$$(\delta ny)_6=-\tfrac{1}{2}\frac{1}{T}\Big\{E_5A^3\sin\psi+E_5B_5A^2xT(2\sin\psi\cos\psi) +E_5B_5^2A\sin\psi(xT)^2+\frac{P_5}{T}A\sin\psi(xP)^2\Big\}. \quad (20)$$

The quantity $B_5 - \frac{b_5 q_1}{q_5'} T$ can be expressed in two forms.

The quantity $\frac{b_5 q_1}{q_5'}$ is the sum of quantities of the form

$$\frac{a_4}{q_3\ q_5\ \ q_1\ q_1} \quad \ldots\ldots\ldots \quad (21)$$

taken from the plane (1) to the central plane considered.

This is a special case of the general theorem, that the apparent distance of any point on an optical system may be obtained as follows :—

The actual distance from any plane in the first medium is divided into parts by the various central planes, each part is multiplied by the refractive index of the medium and divided by the product of the values of q (calculated from any ray) at the bounding planes and the sum gives the corresponding quantity for the apparent distance in the first medium.

A demonstration of the analogous case where the distances are subdivided by the surfaces themselves is given in a Paper by the author.*

Secondly, let the ray parallel to the axis at the height Q in the last medium meet the various central planes in intercepts (of optical length) $Q_1 Q_3 Q_5 \ldots Q$, and let the corresponding intercepts on the image planes in the first medium be $Q_1' Q_3' Q_5' Q'$. Then

$$\frac{Q_5'}{Q_5} = \frac{q_5'}{q_5}.$$

$\frac{Q_5' - Q_1'}{Q} = b_5 P$, where P is calculated for this ray.

The relation between P' and P is given by

$$\frac{T}{S_1} = 1/m = \frac{Q_1}{Q} + \frac{P}{S_1},$$

see equation (10*b*), so that

$$P = T - \frac{Q_1}{Q} S_1.$$

Also $$\frac{q_5' - q_1}{q_1} = -b_5 S_1$$

from Fig. 3, so that

$$\frac{Q_5'}{Q_1} - \frac{q_5'}{q_1} = b_5 P \frac{Q}{Q_1} + b_5 S_1 = b_5 T \frac{Q}{Q_1},$$

* Trans. Opt. Soc., Vol. XVIII., April, 1917.

or $$\left(\frac{Q_5}{Q_1}-\frac{q_5}{q_1}\right)\frac{q_5'}{q_1}=b_5T\frac{Q}{Q_1},$$

i.e., $$\left(\frac{Q_5}{Q_1}-\frac{q_5}{q_1}\right)\frac{q_1}{q_5}=\frac{b_5q_1}{q_5'}T\frac{Q}{Q_1},$$

i.e., $$\frac{b_5q_1}{q_5'}T=\left(\frac{Q_5}{Q_1}\Big/\frac{q_5}{q_1}-\frac{Q_1}{Q}\right).$$

Thus, B_5T may be expressed in either of the two forms

$$\sum\frac{a_4}{\frac{q_3}{q_1}\cdot\frac{q_5}{q}}P \text{ or } \left(\frac{Q_5}{Q}\Big/\frac{q_5}{q_1}-\frac{Q_1}{Q}\right) \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where $q_1 \ldots q_5$ are calculated for the actual object point, and $Q_1 \ldots Q_5Q$ for a ray incident parallel to the axis in the last medium.

In the above it has been assumed that the ray is best specified by its points of crossing the first central plane and the object plane, but in practice it is more convenient to choose the stop plane for the co-ordinates x_1y_1, and in this case we use the same method, except that surface 1 is taken to be a surface of zero power—*i.e.*, no change of refractive index—with its central surface on the stop plane. The analysis is then unaltered.

In actual practice there is little difference in the amount of computation to obtain B_5T from the two expressions given above.

Thus, the resultant aberrational defects may be obtained by summing the effects due to all surfaces and we obtain the five aberrational coefficients

$$\begin{aligned} E&=\Sigma E_6\\ I&=\Sigma E_6B_6\\ K&=\Sigma E_6B_6{}^2\\ G&=K+\Sigma P_5/T\\ H&=\Sigma GB_6 \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

A sample calculation is annexed.

Interpretation of Results.

I. For the terms independent of θ we have

$$\delta nx=-\tfrac{1}{2}\frac{1}{T}EA^3\cos\psi,$$

$$\delta ny=-\tfrac{1}{2}\frac{1}{T}EA^3\sin\psi,$$

so that each circle on the stop plane corresponds to a small circle in the image and the radius varies as A^3. This defect is uniform over the field, and is referred to as Spherical Aberration of the First Order.

II. The coefficient I. gives rise to terms

$$\delta nx = -\tfrac{1}{2}\frac{1}{T} IA^2 \tan\theta \ (\cos 2\psi + 1),$$

$$\delta ny = -\tfrac{1}{2}\frac{1}{T} IA^2 \tan\theta \ (\sin 2\psi),$$

i.e., the effect is proportional to $\frac{1}{T}\tan\theta$ or nx; it also varies as A^2 and to each circular zone in the stop corresponds a double

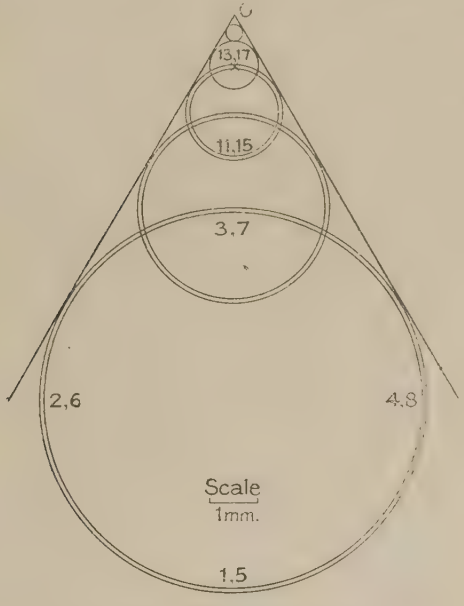

FIG. 4.

circle in the image plane of diameter equal to the distance of its centre from the ideal focal point.

Thus, the defect leads to an unsymmetrical flare, as indicated in Fig. 4, and is called Coma of the First Order.

When spherical aberration is present the two circles no longer co-incide, and they become a double loop.

III. The terms K and G lead to

$$\delta x = -\tfrac{1}{2}T\tan^2\theta A\cos\psi. \quad (2K+G),$$
$$\delta y = -\tfrac{1}{2}T\tan^2\theta A\sin\psi. \quad G,$$

so that the circle on the stop corresponds to an ellipse on the focal plane whose axes are in the ratio of $2K+G : G$; when $K=0$, the ellipse becomes a circle, and the defect being proportional to A is of the same type as that due to an error in focus $= -\tfrac{1}{2}P\tan^2\theta G$. When K is not 0 the effect is that two focal lines are produced at distances from the focal plane

$$= -\tfrac{1}{2}T\tan^2\theta(2K+G),$$

and $$-\tfrac{1}{2}T\tan^2\theta G.$$

K thus determines the astigmatism of the lens. The mean value of $-\tfrac{1}{2}T\tan^2\theta\,(K+G)$ determines the distance of the best average focus from the focal plane. The term $G-K=\Sigma P_5/T$, being independent of separations except in so far as they affect T, is simply calculated, and is often referred to as Petzval's expression.

IV. The coefficient H, being independent of A, must represent a departure of the image from its theoretical position, and is called the distortion; its effect is proportional to the cube of tan θ, and it leads to the defect of unequal magnification for objects of different size, thus making straight lines appear to be curved.

The curvature of any straight line is $-H\tan\theta_1 T$, where θ_1 is the least angular value of θ for points on the straight line.

Formulæ referred to Vertices of Surfaces.

Exactly analogous results are obtained by considering all distances measured from the surface. If the same method of treatment be adopted we obtain the following results :—

Let $$F_1=(n_2-n_0)R_1\,; \quad \frac{B_1B_2}{n_2}=d_2.$$

$$\frac{B_1A_0}{n_0}=u_1\,; \qquad \frac{B_1A_2}{n_2}=v_1.$$

$$U_1=\frac{1}{u_1}\,; \qquad V_1=\frac{1}{v_1}.$$

$$F_1=V_1-U_1\,; \qquad \frac{l_3}{l_1}=1-d_2F_1.$$

$$V_{13}=V_1+\frac{l_3}{l_1}F_3\,; \qquad \frac{l_5}{l_1}=\frac{l_3}{l_1}-d_4V_{13}.$$

$$v_3=\frac{l_3}{l_1}/V_{13}. \qquad \ldots\ldots\ldots \quad (24)$$

The apparent distance d_0 of any point from the first surface is given by

$$\frac{d_0}{\frac{l_0}{l_1}\frac{l_1}{l_1}} = \sum \frac{d_4}{\frac{l_3}{l_1}\frac{l_5}{l_1}},$$

where the summation is taken from the first plane to the plane of the point.

For the case of the centre of curvature (5)

$$\frac{d_0}{\frac{l_0}{l_1}\frac{l_1}{l_1}} = \sum \frac{d_4}{\frac{l_3}{l_1}\frac{l_5}{l_1}} + \frac{r_5}{n_4\frac{l_5}{l_1}\frac{l_6}{l_1}}, \quad . \quad . \quad . \quad . \quad (25)$$

where l_6/l_1 is the intercept at the centre of curvature.

The lateral aberration due to surface 5 is given by

$$\begin{aligned}\delta x = -\tfrac{1}{2}f\,.\,\{&e_5(n\sin A)^3\cos\psi + i_5(n\sin A)^2(1+2\cos^2\psi)\tan\theta \\ &+2k_5\cos\psi(n\sin A)\tan^2\theta + g_5\cos\psi n\sin A\tan^2\theta \\ &+h_5\tan^3\theta\}\,.\end{aligned}$$

$$\begin{aligned}\delta y = -\tfrac{1}{2}f\,.\,\{&e_5(n\sin A)^3\sin\psi + i_5(n\sin A)^2(2\sin\psi\cos\psi)\tan\theta \\ &+g\sin\psi(n\sin A)\tan^2\theta\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)\end{aligned}$$

where n is the refractive index of the final medium in which δx and δy are measured; A, ψ and θ have the same values as in the equations (19) and (20); $f = 1/V$, and

$$e_5 = \frac{l_5}{l_1}\left(\frac{V_{135}}{n_6^2} - \frac{V_{13}}{n_4^2}\right)\left\{\frac{1}{D_5'F}\right\}^2,$$

$$i_5 = D_5 e_5 \qquad k_5 = D_5 i_5,$$

$$g_5 - k_5 = P_5/V = R_5\left(\frac{1}{n_4} - \frac{1}{n_6}\right)/V,$$

$$h_5 = D_5 g_5.$$

$$\frac{1}{D_5'V} = +\frac{n_6 l_6}{r_5 l_1}\,\frac{1}{V} = -\left(\frac{V_{135}}{n_6} - \frac{V_{13}}{n_4}\right)\Big/\left(\frac{1}{n_6} - \frac{1}{n_4}\right)V,$$

and

$$D_5V = \frac{l_1}{l_5}D_5'V + \sum_{-2}^{4}\frac{d_4V}{\frac{l_3}{l_1}\frac{l_5}{l_1}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The value of D_5V may also be expressed as

$$\frac{l_1}{l_5}D_5'V + \left(\frac{L_5}{L}\Big/\frac{l_5}{l_1} - \frac{L_1}{L}\right), \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where L_1, L_5 L are quantities similar to l_1, l_3, l_5 calculated for the ray incident parallel to the axis and in the reversed direction.

By either of the methods given already the Aberrations of the First Order can be calculated for any definite object position. When the lens system has to be used for different magnifications it is preferable to make use of the method of the characteristic function, either calculating the coefficient directly or using the above results to deduce the coefficients of the characteristic function.

The annexed calculation shows the practical application of the formulæ given above; the logarithms are omitted when the calculation is quite straightforward, but inserted where the arrangement of the calculations is important.

[*Added February* 9.

The following method of expressing the value of p in terms of the co-ordinates of the rays is often useful :—

To the approximation considered p_5 may be replaced by the optical length of the intercept on the central plane of 5 by the ray.

Consider the images in medium 6 of the object plane (0) and the stop plane (1), and let the ray considered meet these planes in P_6, Q_6, and the central plane in K_6. Let the corresponding axial points be A_6 and B_6. Let this ray meet the object plane in $[x_0, y_0]$ and the stop plane in $[x_1, y_1]$, then the ray through x_0, y_0 on the object plane and o, o on the stop plane will pass through P_6 and B_6 to the first approximation, while the ray through o, o on the object plane and $[x_1, y_1]$ on the stop plane will pass through A_6 and Q_6 to the same approximation.

Let these rays meet the central plane in K_6' and K_6'',

then $$C_5K_6=C_5K_6'+C_5K_6''.$$

If q_1, q_3, q_5, &c., be calculated for the paraxial ray through o, o meeting the stop at q_1 from the axis, we have

$$[C_5K_6']=[x_1, y_1]\frac{q_5}{q_1}.$$

Aberrations of Photographic Lens.

Object at infinity. $S=O$.

$r_1=-0{\cdot}128935$	$d_0=0{\cdot}01583$	$n_0=1$	log.
$r_3=-0{\cdot}049597$	$d_2=0{\cdot}01277$	$n_2=1{\cdot}51170$	$=0{\cdot}179466$
$r_5=+0{\cdot}196423$	$d_4=0{\cdot}00664$	$n_4=1{\cdot}54780$	$=0{\cdot}189715$
$r_7=-0{\cdot}126663$	$d_6=0{\cdot}02114$	$n_6=1{\cdot}61250$	$=0{\cdot}207500$
		$n_8=1$	

$P_1=\dfrac{n_2-n_0}{n_2n_0}\dfrac{1}{r_1}=-2{\cdot}62468$ $\quad a_0=(d_0+r_1)\,.\,n_0=-0{\cdot}113135$

$P_3=\dfrac{n_4-n_2}{n_4n_2}\dfrac{1}{r_3}=-0{\cdot}311079$ $\quad a_2=n_2(-r_1+d_2+r_3)=+{\cdot}0139285$

$P_5=\dfrac{n_6-n_4}{n_6n_4}\dfrac{1}{r_5}=+0{\cdot}131976$ $\quad a_4=n_4(-r_3+d_4+r_5)=+0{\cdot}391067$

$P_7=\dfrac{n_8-n_6}{n_8n_6}\dfrac{1}{r_7}=+2{\cdot}99887$ $\quad a_6=n_6(-r_5+d_6+r_7)=-0{\cdot}486889$

	Log.	No.			No.
P_1q_1.....	0·419077−	−2·62468	1·000$=q_1$	$P_1/T_{1\cdot3\cdot5\cdot7}$	−2·62460
$-a_2$	$\bar{1}$·143905−			$P_3/T_{1\cdot3\cdot5\cdot7}$	−0·31107
	$\bar{1}$·562982+		=0·36558	$P_5/T_{1\cdot3\cdot5\cdot7}$	+0·13197
	0·135317		1·36558$=q_3$	$P_7/T_{1\cdot3\cdot5\cdot7}$	+2·99878
P_3.....	$\bar{1}$·492871−				
P_3q_3	$\bar{1}$·628188−	−0·42480		a_0	−0·113135
	0·484226−	−3·04948$=T_{1\cdot3}$		a_2T/q_1q_3	+0·101994
$-a_4$	$\bar{1}$·592251−			Σ ,,	−0·011141
	0·076477+		=1·19255	a_4T/q_3q_5	+0·111944
	0·407922+		2·55813$=q_5$	Σ ,,	+0·100803
P_5	1·120496+			a_6T/q_5q_7	−0·153763
P_5q_5	1·528418+	+0·33761		Σ ,,	−0·052960
$T_{1\,3\cdot5}$ ...	0·433269−	−2·71187$=T_{1\cdot3\cdot5}$			
$-a_6$	$\bar{1}$·687429+				
	0·120698−		=−1·32035		
	0·092643+		+1·23778$=q_7$		
P_7	0·476957				
P_7q_7.....	0·569600+	+3·71190			
	0·000013	+1 00003$=T_{1\cdot3\cdot5\cdot7}$			

q_1 is taken as 1, so that the quantities $\dfrac{q_3}{q_1}\ \dfrac{q_5}{q_1}$, &c., may be denoted by q_3, q_5, &c.

—	Log.	No.	Log.	No.	Log.	No.	Log.	No.	Aberration coefficients for complete lens.
P_1q_1	0·419077−	...	$\bar{1}$·628188−	...	$\bar{1}$·528418+	...	0·569600+	...	...
$(Pq)^1$	1·257231−	...	$\bar{2}$·884564−	...	$\bar{2}$·585254+	...	1·708800+	...	...
q_1	0·000000	...	0·135317+	...	0·407922+	...	0·092643+	...	...
$1/n_2-n_0$	0·290985+	...	1·442493+	...	1·189096 +	...	0·212894−	...	...
n_2n_0/n_2-n_0 .	0·470451+	...	1·811674+	...	1·586311+	...	0·420394−	...	...
	2·018667−	−104·392	2·274048−	−187·953	1·768583+	+58·6926	2·434731+	+272·102	...
P_1q_1	0·419077−	...	1·628188−	...	$\bar{1}$·528418+	...	0·569600+	...	...
S_1.....	...	0	0·419077−	...	0·484226−	...	0·433269−	...	...
T_1	0·419077	...	0·484226−	...	0·433269−	...	0·000013 +	...	...
q_1	...	...	0·135317+	...	0·407922+	...	0·092643+	...	...
	...	0	0·666808−	−4·64310	0·853835+	+7·1422	1·095525−	−12·460	...
E	2·018667−	−104·392	2·263186−	−183·310	1·712232+	+51·5504	2·454177+	+284·562	+48·411
$\Sigma a/q\,.\,q$	$\bar{1}$·053597−	...	2·046924−	...	1·003473+	...	$\bar{2}$·723948−	...	...
I	$\bar{1}$·072264+	+11·8104	0·310110+	+2·04226	0·715705+	+5·1964	1·178125−	−15·0704	+3·9787
	$\bar{1}$·053597−	...	$\bar{2}$·046924−	...	$\bar{1}$·003473+	...	$\bar{2}$·723948 −	...	...
K	0·125861−	−1·33617	$\bar{2}$·357034−	−0·02275	$\bar{1}$·719778+	+0·52381	1·902073+	+0·79813	−0·03698
P_1/T	...	−2·62460	...	−0·31107	...	+0·13197	...	+2·99878	...
G	0·597780−	−3·96077	$\bar{1}$·523512−	−0·33382	$\bar{1}$·816758+	+0·65578	0·579430+	+3·79691	−0·15810
	$\bar{1}$·053597 −	...	$\bar{2}$·046924−	...	$\bar{1}$·003473+	...	$\bar{2}$·723948−	...	...
H	$\bar{1}$·651377+	+0·44810	$\bar{3}$·570436+	+0·003719	$\bar{2}$·820231+	+0·066105	$\bar{1}$·303378−	−0·20108	+0·31684

If q_0', q_3', q_5', &c., be calculated for the paraxial ray through o, o, meeting the object plane at q_0' and the central planes at q_3', q_5', &c., we have

$$[C_5K_6'']=[x_0, y_0]\frac{q_5'}{q_0'},$$

and

$$[C_5K_6]=[x_1, y_1]\frac{q_5}{q_1}+[x_0, y_0]\frac{q_5'}{q_0'}.$$

And this quantity can be substituted for p_5 in equation 16.

ABSTRACT.

The Paper describes approximate methods of treatment of the first order aberrations of a centred optical system. Two methods are used, one primarily suited to the case where the separation of the surface is small, and the other more suited for use where the separations vitally affect the design.

The aberrational defects are expressed as lateral aberrations—*i.e.*, as defects measured in the focal plane of the system. The procedure adopted is to express the aberrational defect of a single surface in terms of the constants of the surface, and the perpendicular distance of the ray considered from the centre of curvature of the surface. The value of this perpendicular can be expressed in terms of the co-ordinates of the ray in any chosen medium, and thus the aberration due to each surface can be expressed in terms of the co-ordinates of the chosen ray, in such a way that the aberrations of the individual surfaces can be summed.

DISCUSSION.

Mr. T. Smith congratulated the author on his Paper, which he considered an interesting and valuable addition to the existing discussions on optical aberrations. He had himself used a similar reference system in dealing with this subject, but in place of a partially geometrical discussion had preferred to use a purely analytical method. The geometrical treatment was obviously advantageous in giving an insight into the mysteries of aberration to students, but it was very difficult to be certain that every contributory cause had been considered. The results stated in the Paper were correct, but he was not clear that one possible contributory cause, which happened to this approximation to give a zero contribution, had received adequate consideration. If in a system of refracting surfaces at some intermediate theoretical image plane an aberration δy were present, the corresponding effect in the final image plane should not be assumed to be $m\delta y$, where m is the first order magnification which the image subsequently undergoes. The exact co-ordinates to the second approximation $y+\delta y$, and the exact magnification to the same order $m+\delta m$ should be considered, so that the final co-ordinates to this order are $my+y\delta m+m\delta y$. It can be shown that when all the surfaces are taken into consideration the contribution of the type $y\delta m$ is zero. He thought that the Paper could with advantage consider this point more fully. An interesting question which arose in connection with the analysis of refraction into displacements according to the ordinary laws of geometrical optics with aberrations superposed, related to the limits within which such a procedure was justifiable. The ordinary

laws could only be taken as a possible first approximation for all rays for refractive indices exceeding 3. Instead of an incident angle in air of a right angle, the first approximation may only be used up to an incident angle of $\cos^{-1}\frac{3-\mu}{2\sqrt{2}}$. As the index falls in value the range over which the approximation holds becomes less, and in the limiting case, when $\mu=1$, the range is from 0° to 45° compared with the range 0° to 90° over which refraction takes place.

Prof. LEES said that these methods appeared to start off from the approximate assumption that the image of a plane was a plane. Did not this lead to errors comparable with those which were being dealt with ?

Mr. CHALMERS replied as follows : The criticisms made by Mr. Smith are not, I think, justified, because the terms to which he refers do not arise in the method of treatment adopted. In transferring the aberration to the last medium it is permissible to multiply by the paraxial magnification since the actual point of crossing the object plane is transferred to the image plane by multiplying its co-ordinates by the paraxial magnification, and adding aberrational defects for this surface for the actual ray. This aberrational defect is expressed in terms of the modified values of the co-ordinates, but as the modifying terms are multiplied by the small aberrational terms this introduces terms of higher order only. If it be necessary to consider terms of higher order, it would be necessary to take them into account in expressing values of the p's in terms of the co-ordinates in the initial medium.

Mr. Smith's remarks on limits to which approximation could be applied are very interesting : they show that approximations are justified in practically all cases which arise, except for a few cases of refraction at plane surfaces, and in this case the term he quotes is multiplied by a zero curvature.

IX. *Note on the Use of Approximate Methods in Obtaining Constructional Data for Telescope Objectives.* *By* T. Smith, *B.A.* (*From the National Physical Laboratory.*)

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In earlier communications* the author has described a method of calculating cemented telescope objectives when thicknesses are entirely neglected. The effects of the thicknesses of actual objectives on the aberrations are distinctly appreciable in spite of the small values of the thicknesses in comparison with the focal length of the objective. This, however, so far from invalidating the calculations, is an essential feature of the method employed. The reason is that the second order aberrations are always negative, while the introduction of thicknesses causes the first order aberrations to rise from zero to some positive value. To obtain a satisfactory objective a balance must be struck between a positive first order aberration and a negative second order aberration, and the greater the relative aperture at which the objective is to be used the greater must be the amount of first order aberration introduced to effect the most satisfactory compromise. The larger the aperture, however, the larger must the thicknesses of the lenses be, so that without altering the curvatures of the surfaces increasing amounts of first order aberration will be automatically secured. It is a fortunate coincidence that the amounts of aberration introduced by augmenting the thicknesses are of the order required to give a satisfactory objective of the aperture these thicknesses will yield. This result, of course, only holds within limits; but these limits happen to cover the region within which small objectives are normally constructed. Many computers do not appear to be aware of this automatic compensation, but have abandoned the use of approximate methods in calculating simple objectives in favour of trigonometrical ray tracing. They have probably in their approximate methods taken account of the thicknesses to be given to the lenses, and solved the equations for the removal of first order aberrations with these additional terms present. Such a method does undoubtedly lead to objectives which require appreciable correction. The simpler method, on the other hand, yields objectives which require no alteration or trigonometrical verification, provided the conditions are not

* Proc. Phys, Soc. XXVII., p. 485; XXVIII., p. 220; XXX., p. 31.

quite abnormal. It must be understood that this observation applies only to the particular class of objective now considered, though extensions of the application of approximate methods as satisfactory final methods of calculation are to be expected as a consequence of the systematic exploration of higher order aberrations in other types of optical instrument.

The automatic compensation for thicknesses in the case of telescope objectives is of sufficient interest and importance to merit theoretical investigation, and that is the object of this note. A simple demonstration will show that the compensation may hold for a considerable range of apertures. Consider any thin objective which has such curvatures that the first order spherical aberration is zero. Without altering the curvatures of the surfaces, let small but finite thicknesses be given to the various component lenses. A series of objectives of different thicknesses can be imagined in which the ratios of the thicknesses of the various component lenses to one another are constants throughout the series. A single variable, t, proportional to the thicknesses of a typical member of the series, will then suffice to determine that member. When t is zero the first order spherical aberration coefficient is zero. When t is not zero but small, the first order spherical aberration coefficient will in general be proportional to t; let it be denoted by αt, where α is a constant for all lenses of the series considered. As has been pointed out above, α is essentially positive. The second order spherical aberration coefficient is for these objectives negative, and, when the lens is thin, may be denoted by $-\beta$. When thicknesses are introduced there will be a correction to be added to β containing t as a factor; but, as β is a large positive number, the thickness correction to it is negligible, and the second order coefficient may thus be taken to be equal to $-\beta$ for all lenses of the series. If the third and higher order aberrations may be neglected, the total longitudinal aberration of the typical objective is

$$\alpha t P^2 - \beta P^4,$$

where P is the relative semi-aperture for the ray considered. Now, if Q is the maximum semi-aperture attainable in this objective, t is proportional to Q^2. The aperture P for which correction is desired is a constant fraction of the full aperture Q. Thus correction should be effected for the aperture P where $P^2=\gamma t$, γ being positive and constant for all members of the

series. On substituting this value for P^2, the longitudinal aberration is seen to be

$$(\alpha-\beta\gamma)\gamma t^2,$$

and this will vanish for all members of the series if

$$\alpha=\beta\gamma.$$

This result shows that the compensation may be automatically effected over an appreciable range of apertures. The extent of the range is dependent, among other factors, on the third and higher order aberrations.

If the thin objective is corrected for first order coma, a similar argument shows that there may be automatic compensation for this aberration also. It will not, however, ordinarily be the case that the spherical aberration and coma are corrected in this way for the same aperture. As a rule the effect of thicknesses on coma can be safely neglected. The conclusions to which this argument leads may be put into the general form :

" If a thin lens is free from first order spherical aberration or coma, the introduction of thicknesses without any alteration in the curvatures of the surfaces yields objectives which are corrected for a zone bearing a constant ratio to the maximum aperture obtainable with the thicknesses introduced, provided effects proportional to the sixth and higher powers of the maximum aperture may be neglected."

The problem may now be investigated in some detail. The case of a cemented doublet may be taken as an illustration. It would be easy to work out the effects of the thicknesses on such a system by taking the curvatures of the thin objective, inserting thicknesses and calculating the aberrations of the new system from the fundamental formulæ. This course involves the introduction of more variables than are necessary for the discussion of the problem, and an alternative method is therefore followed.

The spherical aberration σ and coma σ' of any system are expressed in terms of the six aberration coefficients, A, B, C, ϖ, B', A', by the formulæ

$$\sigma=A-m(4B+1)+2m^2(3C+\varpi)-m^3(4B'+1)+m^4A',$$

and

$$\sigma'=A-m(3B+1)+m^2(3C+\varpi)-m^3B' \\ -s\{B-m(3C+\varpi)+m^2(3B'+1)-m^3A'\},$$

where m is the magnification for the object and s the magnifi-

cation for the aperture stop. When the objective is "thin" the relations

$$A-2B-1+C=B-2C-\varpi+B'=C-2B'-1+A'=0$$

are satisfied, so that

$$4\sigma=(1-m)^4\{4C+2\varpi+1-4(B'-B)M+(2\varpi+3)M^2\}$$

and

$$4\sigma'=(1-m)^3(1-s)\{4C+2\varpi+1-(B'-B)(S+3M)+\varpi M(S+M)+M(2S+M)\},$$

where

$$M=\frac{1+m}{1-m},\ S=\frac{1+s}{1-s}.$$

For the thin objective the conditions for freedom from spherical aberration and coma thus involve

$$4C+2\varpi+1=(5+2\varpi)M^2$$

and

$$B'-B=(2+\varpi)M.$$

The actual thick objective may be regarded as a combination of three systems; the first being a meniscus single lens of the same material as the front component of the thin doublet, with a specified thickness, and having the curvature of each surface equal to that of the first surface of the doublet; the second the thin doublet; the third a single meniscus lens of the same glass as the back component of the doublet, with a given thickness, and each surface equal in curvature to that of the last surface of the doublet. If suffixes are used to distinguish the components and the magnifications at which they are operating, the laws for the addition of the aberrations of a series of systems may be expressed in the form

$$f_{1,n}\sigma_{1,n}=\sum_1^n f_\lambda\sigma_\lambda m^4{}_{\lambda+1,n}$$

$$f_{1,n}\sigma'_{1,n}=\sum_1^n f_\lambda\sigma'_\lambda m^3{}_{\lambda+1,n}\,s_{\lambda+1,n},$$

where f is the focal length of the lens or element determined by its suffix.

Let κ denote the power of the first surface of the thin doublet, the refractive index being μ. The first component is the meniscus lens of the same glass of thickness t and surfaces of powers κ and $-\kappa$ respectively. The general formulæ for

the aberration coefficients at once give for the first component

$$A(\mu-1)^2\kappa^2t^2=(\mu+1)^2-2\kappa t(\mu^2+\mu+1)+\kappa^2t^2(\mu+1)^2-\kappa^3t^3\mu,$$
$$B(\mu-1)^2\kappa^2t^2=(\mu+1)^2-\kappa t(\mu^2+\mu+1)+\kappa^2t^2\mu,$$
$$C(\mu-1)^2\kappa^2t^2=(\mu+1)^2,$$
$$B'(\mu-1)^2\kappa^2t^2=(\mu+1)^2+\kappa t(\mu^2+\mu+1)+\kappa^2t^2\mu,$$
$$A'(\mu-1)^2\kappa^2t^2=(\mu+1)^2+2\kappa t(\mu^2+\mu+1)+\kappa^2t^2(\mu+1)^2+\kappa^3t^3\mu,$$
$$\varpi=0.$$

In the last component the power of the first surface has the opposite sign to that of the last surface of the doublet. Similar formulæ will thus apply to this component if accents are added and the sign of κ is changed, if the back lens of the thin doublet is of refractive index μ' with the last surface of power κ', and t' is the thickness to be introduced.

The powers of the three components of which the thick lens is supposed to be composed are

$$\kappa_1=\kappa^2t/\mu, \qquad \kappa_2=1, \qquad \kappa_3=\kappa'^2t'/\mu',$$

with

$$\kappa_{1,2}=1-\kappa(1-\kappa)t/\mu, \qquad \kappa_{2,3}=1-\kappa'(1-\kappa')t'/\mu',$$

and

$$\kappa_{1,3}=1-\kappa(1-\kappa)t/\mu-\kappa'(1-\kappa')t'/\mu'+\kappa\kappa'(1-\kappa-\kappa')tt'/\mu\mu'.$$

The partial magnifications are given in terms of m, the magnification for the complete system by

$$m_1=\frac{\kappa_{1,3}m}{\kappa_1+\kappa_{2,3}m}, \qquad m_2=\frac{\kappa_1+\kappa_{2,3}m}{\kappa_{1,2}+\kappa_3m}, \qquad m_3=\frac{\kappa_{1,2}+\kappa_3m}{\kappa_{1,3}}.$$

If only the first powers of t and t' are retained these may be written

$$m_1=1-\frac{\kappa t}{\mu}p,$$
$$m_2=m\left(1+\frac{\kappa t}{\mu}p-\frac{\kappa't'}{\mu'}q\right),$$

and

$$m_3=1+\frac{\kappa't'}{\mu'}q,$$

where

$$p=1-\kappa+\kappa/m,$$

and

$$q=1-\kappa'+\kappa'm.$$

Now from the previous formulæ

$$\sigma_1=\frac{1}{(\mu-1)^2\kappa^2t^2}\,[(\mu+1)^2(1-m_1)^4-2\kappa t(\mu^2+\mu+1)(1-m_1)^2$$
$$(1-m_1^2)+\kappa^2t^2(\mu+1)^2(1-m_1)(1-m_1^3)-\kappa^3t^3\mu(1-m_1^4]$$
$$=\frac{\kappa^2t^2p(p-\mu)}{(\mu-1)^2\mu^4}\,[(\mu+1)^2p^2-\mu(3\mu^2+2\mu+3)p+4\mu^3],$$

and similarly

$$\sigma_3=\frac{\kappa'^2t'^2q(q-\mu')}{(\mu'-1)^2\mu'^4}\,[(\mu'+1)^2q^2-\mu'(3\mu'^2+2\mu'+3)q+4\mu'^3],$$

when the lowest powers of t and t' only are retained. The direct contribution of the first component to $f_{1,3}\sigma_{1,3}$ is, therefore, to the order required

$$\frac{tm^4p(p-\mu)}{(\mu-1)^2\mu^3}\,[(\mu+1)^2p^2-\mu(3\mu^2+2\mu+3)p+4\mu^3],$$

and that of the last component is

$$\frac{t'q(q-\mu')}{(\mu'-1)^2\mu'^3}\,[(\mu'+1)^2q^2-\mu'(3\mu'^2+2\mu'+3)q+4\mu'^3].$$

The second component has been calculated to be free from aberration for the magnification m, but it is now working at the different magnification m_2. If sigmas without suffixes relate to the thin lens as calculated, the condition $\sigma=0$ requires the coefficients of the doublet to have the values

$$B'-B=(2+\varpi)M-\frac{2\sigma'}{(s-m)(1-m)^2},$$

and

$$4C+2\varpi+1=(5+2\varpi)M^2-\frac{8\sigma'M}{(s-m)(1-m)^2},$$

σ' being the coma coefficient of the thin objective.

For the magnification m_2 the spherical aberration of the thin objective is, therefore, given by

$$\sigma_2=m\left(\frac{\kappa t}{\mu}p-\frac{\kappa't'}{\mu'}q\right)\left(\frac{4\sigma'}{s-m}+m^2-1\right)$$

since

$$M_2=\frac{1+m_2}{1-m_2}=M+\frac{2m}{(1-m)^2}\left(\frac{\kappa t}{\mu}p-\frac{\kappa't'}{\mu'}q\right).$$

The value of the spherical aberration coefficient for the lens

with thicknesses introduced is thus, to the first powers of the thicknesses,

$$\sigma_{1,3}=-\frac{tm^4p(p-\mu)}{(\mu-1)^2\mu^3}[(\mu+1)^2p^2-\mu(3\mu^2+2\mu+3)p+4\mu^3]$$
$$+m\left(\frac{\kappa t}{\mu}p-\frac{\kappa' t'}{\mu'}q\right)\left(\frac{4\sigma'}{s-m}+m^2-1\right)$$
$$+\frac{t'q(q-\mu')}{(\mu'-1)^2\mu'^3}[(\mu'+1)^2q^2-\mu'(3\mu'^2+2\mu'+3)q+4\mu'^3].$$

The most important case is $m=0$. The value of mp is then equal to κ, and

$$\sigma_{1,3}=\frac{t\kappa^4(\mu+1)^2}{(\mu-1)^2\mu^3}+\frac{t\kappa^2}{\mu}\left(\frac{4\sigma'}{s}-1\right)$$
$$+\frac{t'q(q-\mu')}{(\mu'-1)^2\mu'^3}[(\mu'+1)^2q^2-\mu'(3\mu'^2+2\mu'+3)q+4\mu'^3].$$

Take as a numerical example the ordinary form of objective composed of hard crown and dense flint with the former leading.* The last surface is flat, so that q is unity and the contribution of the third element to $\sigma_{1,3}$ is thus

$$\frac{-t'(\mu'^2-1)}{\mu'^3},$$

the ordinary formula for a thick plate. With the glass proposed its value is very approximately $-\cdot38t'$. The value of κ is about 1·25,* and the contribution of the first element is, therefore, approximately equal to $16\cdot5t$. When $s=1$, $\sigma'=-1\cdot47$,* so the contribution of the doublet is $-7\cdot1t$. As a rule t' is about half as large as t, so the total value is approximately $9\cdot2t$. The second order spherical aberration coefficient for the thin lens is about -107, so to the second order the spherical aberration with crown thickness t is

$$\tfrac{1}{2}\times9\cdot2tP^2-\tfrac{1}{8}\times107P^4,$$

where P is the relative semi-aperture. The zone for which the aberration is corrected is thus given by $P^2=0\cdot34t$. The maximum semi-aperture the crown lens can have for this thickness is Q where $Q^2=0\cdot42t$. Thus $P=0\cdot9Q$ approximately, a value quite satisfactory for most purposes. The neglected

* See "Constructional Data of Small Telescope Objectives." Smith & Cheshire. Pp. 8 and 9.

aberrations tend to reduce this value somewhat when the aperture is large, the correction thus remaining excellent over a long range.

As the refractive index of the crown glass approaches that of the flint, κ becomes smaller and σ', though still negative, becomes smaller in absolute amount. The total contribution of the elements remains positive, but falls in value. The second order aberration coefficient also becomes less in absolute amount, and the tendency for the correction to be automatically secured for a reasonable zone compared with the full aperture attainable remains.

ABSTRACT.

The Paper discusses the reason why satisfactory telescope objectives are obtained by neglecting thicknesses and solving for freedom from first order aberrations. It is shown that the introduction of thicknesses into such an objective without any alteration in the curvatures of the surfaces yields a lens corrected for aberration for a zone which is a constant fraction of the full aperture obtainable. For objectives of the usual type this zone is very approximately the one that would be selected for correction to obtain the most favourable balance between first and second order aberrations. It follows that objectives calculated from first order formulæ in which thicknesses are neglected do not require trigonometrical verification or correction unless the conditions are very abnormal.

DISCUSSION.

Prof. J. W. Nicholson said that the author had brought out an important point in this Paper. There was an idea prevalent in many quarters that the thin lens was simply an abstraction of no practical importance, but apparently this was by no means the case. He had been familiar in a general way with this compensation of first and second order aberrations, but had never seen it mentioned in any publication. Were there any convenient circumstances in which both coma and spherical aberration are simultaneously compensated for the same aperture ?

Prof. C. H. Lees asked how the magnitude of the residual aberration or the shape of the aberration curve altered when the zone of correction was altered by changing the index of the crown glass.

The Author, in reply to Prof. Nicholson, said he had not worked out numerical cases for coma, though the formulæ were quite simple. In practice, it is usually found that in the absence of spherical aberration if the coma is approximately corrected the result is satisfactory. He had no doubt, however, that with triple objectives both coma and spherical aberrration could be compensated for the same aperture. In reply to the President, generally speaking, with very low indices of the crown the second order aberration is large, while with indices nearer the flint it is smaller. The ratio of extreme cases might be about 5 to 1.

X.—*A Suggestion as to the Origin of Spectral Series.* By H. Stanley Allen, *M.A.*, *D.Sc.*

Received January 25, 1918.

Consider an electron (charge e, mass m) moving in a circular orbit of radius r about the centre of the core of an atom. Neglecting for the present any effect due to a magnetic field, the equation of motion is

$$mr\omega^2=Ee/r^2,$$

or
$$mr^3\omega^2=Ee, \quad \ldots\ldots\ldots \quad (1)$$

where ω is the angular velocity, and E the charge attracting the electron.

We now assume that the principle of the constancy of angular momentum may be applied to the *total* angular momentum of the electron and a certain part of the core bearing a special relation to the electron. This angular momentum is measured with regard to the axis of the core, which provides a line of reference about which the motion takes place. The difference between this assumption and that made by Bohr must be noticed. Bohr assumes that the electron and the nucleus rotate round the common centre of gravity, and the total angular momentum of electron and nucleus is then $\tau h/2\pi$ where τ is an integer and h is Planck's constant. The new assumption gives

$$mr^2\omega \mp I\,\Omega=\tau h/2\pi,$$

or
$$mr^2\omega=(\tau \pm \mu)h/2\pi, \quad \ldots\ldots\ldots \quad (2)$$

where
$$\mu=2\pi I\,\Omega/h \quad \ldots\ldots\ldots \quad (3)$$

Here I denotes the moment of inertia, Ω the angular velocity of part of the core.

Thus μ may be regarded as proportional to the angular momentum of a definite portion of the core.* The assumption made is admittedly an arbitrary one, but so is any assumption as to the constancy of angular momentum. This constancy may be regarded as a condition imposed upon the motion of the electron by the "binding" of the tube of force connecting it to the core, but there is something more to be said for the present point of view inasmuch as the connection between electron and core is now partly dependent on the magnetic field, which is regarded as imparting stability to the electronic

* H. S. Allen, Phil. Mag., Vol. XXIX., p. 714, 1915.

system. Thus it is appropriate to take the magnetic axis of the core as the axis for calculating the angular momentum.

From equations (1) and (2) we find as the values of r and ω,

$$r=\frac{(\tau\pm\mu)^2h^2}{4\pi^2mEe}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$\omega=\frac{8\pi^3(Ee)^2m}{(\tau\pm\mu)^3h^3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The energy, W, necessary to remove the electron to an infinite distance is given by

$$W=\frac{Ee}{2r}=\frac{2\pi^2m(Ee)^2}{(\tau\pm\mu)^2h^2}. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

In the passage from one "stationary" state to another monochromatic radiation is supposed to be emitted, and the frequency, ν, is determined by the relation

$$h\nu=\delta W=W_2-W_1$$

$$=\frac{2\pi^2m(Ee)^2}{h^2}\left(\frac{1}{(\tau_2\pm\mu_2)^2}-\frac{1}{(\tau_1\pm\mu_1)^2}\right). \quad (7)$$

Putting $E=ke$, and $\frac{2\pi^2me^4}{h^3}=\nu_0$, the equation may be written

$$\nu=k^2\nu_0\left(\frac{1}{(\tau_2\pm\mu_2)^2}-\frac{1}{(\tau_1\pm\mu_1)^2}\right) \quad . \quad . \quad . \quad . \quad (8)$$

Here ν_0 is the frequency corresponding to Rydberg's constant. Using wave numbers instead of frequencies, we obtain

$$N=k^2N_0\left(\frac{1}{(\tau_2\pm\mu_2)^2}-\frac{1}{(\tau_1\pm\mu_1)^2}\right). \quad . \quad . \quad . \quad (9)$$

Rydberg's constant, N_0, appears here as a universal constant having exactly the same value for all elements. This is in agreement with the conclusion of Nicholson* from a critical examination of the arc spectrum of helium, where the constant has the same value as was obtained by Curtis for hydrogen. It has been suggested by Bohr† that the series of lines observed by Fowler and Evans, and attributed to helium, require a slightly different value for the constant, but this cannot be regarded as definitely established.

In ordinary spectra the numerical factor k must be taken as unity. When $k=2$, the Rydberg constant is multiplied

* Nicholson, Proc. Roy. Soc., A, Vol. XCI., p. 255, 1915.
† Bohr, "Nature," Vol. XCII., p. 231, 1913.

by 1, and the series give the enhanced lines in spark spectra, as shown by Fowler.*

Equation (9) represents Rydberg's formula for spectral series, the quantity μ being termed the "phase" of the series. It is important to notice that in the observed series μ_1 and μ_2 are not equal to one another. If the present interpretation be correct this implies that the angular momentum of the part of the core associated with the electron is different in the two types of state. This may arise from an actual change in the angular momentum of a particular part of the core, or it may indicate that the electron becomes attached to a different part of the core. *Whatever interpretation be given to the phase, the two types of state concerned must be in some way different from one another.*

In an earlier paper† I have shown that when the core is regarded as equivalent to a magnet of moment M a formula of the type given by Ritz can be obtained for the frequency. This may be written

$$N=k^2N_0\left\{\frac{1}{\left[\sigma_2+\frac{B}{\sigma_2^2}\right]^2}-\frac{1}{\left[\sigma_1+\frac{B}{\sigma_1^2}\right]^2}\right\}, \quad \ldots \quad (10)$$

where

$$B=\frac{16\pi^3mMEe^2}{h^3} \quad \ldots \quad (11)$$

Here σ was regarded as an exact integer, but if we now adopt the suggestion that $\sigma=\tau+\mu$, where τ is integral, the formula becomes identical with Ritz's modification of Rydberg's formula. It must be pointed out, however, that the values of B deduced from observation are, in general, too large to be explained as due entirely to the action of the magnetic field of the core.

In the foregoing pages it has been assumed that in the case of a terrestrial atom, as distinguished from the simpler atoms of the nebulæ, the core is large enough to be able to produce an appreciable magnetic field. To say that the core cannot produce such a magnetic field on account of the small size of the nucleus is to beg the question at issue, for the smallness of the nucleus was first deduced by considering only electrostatic forces, whilst ignoring the possible effect of magnetic forces, in the scattering of α particles. The second reason

* Fowler, Phil. Trans. Roy. Soc., A, Vol. CCXIV., p. 225, 1914.

† Phil. Mag., Vol. XXIX., p. 40, 1914.

for postulating a small nucleus is in order to account for the mass of the atom as purely electromagnetic in origin. But this assumes that the nucleus is *simple* and not *complex*. The phenomena of radio-activity point to the presence of α and β particles in the core as separate entities. It is only the ultimate positive entities (perhaps the hypothetical positive electrons of Prof. Nicholson) contained in the core, which need be considered very small in order to give the necessary inertia.

ABSTRACT.

The note gives a development of an idea put forward in an earlier Paper, describing an atomic model with a magnetic core. It is assumed that the principle of the constancy of angular momentum may be applied to the *total* angular momentum of the electron, and a certain part of the core bearing a special relation to the electron. On the lines of Bohr's theory this leads to an expression for the oscillation frequency, which is similar to Rydberg's formula, and contains a constant which is the same for all elements. The "phase" μ of a "sequence" is regarded as proportional to the angular momentum of a definite portion of the core. In observed series the phases of the two sequences are not equal to one another; consequently, whatever interpretation be given to the phase, the two types of state concerned must be in some way different from one another. When the magnetic field of the core is taken into account, a formula is obtained which is identical with that of Ritz. An explanation of the series of enhanced lines in spark spectra is also suggested.

DISCUSSION.

Prof. J. W. Nicholson thought the fundamental assumption in the Paper seemed a natural one. Bohr's theory, in the case of hydrogen, assumed the angular momentum to be proportional to the integers in Balmer's series. In the case of more complex atoms we were compelled to assume that $\tau+\mu$ is the quantity to which the angular momentum is proportional. To consider this extra angular momentum to be attached to the nucleus is, of course, another assumption. It is equivalent to attaching a physical significance to tubes of force, which Sir J. J. Thomson has always tried to do. An interesting feature of the theory was that it gave the series of enhanced lines of spark spectra, and the physical interpretation of $k=2$, viz., that these are due to atoms with another electron torn off, seemed very feasible. As was pointed out in the Paper, any theory involving stationary states must allow for different types of stationary state—*i.e.*, different values of μ_1 and μ_2. The considerations by which Rutherford deduced the smallness of the nucleus were founded on the mathematical work of Darwin, and, on the assumption that the nucleus is wholly electrical are faultless. The fact that the nucleus might be magnetic was pointed out by Hicks, who showed that the scattering of α particles by atoms might be accounted for by magnetons rather than electrons.

Prof. G. W. O. Howe asked what was the reason for applying the word "phase" to μ. To an electrician phase had a very different meaning. What was the significance of the initial equations with which the Paper started out?

Prof. LEES asked if the atom suggested by Dr. Allen would give the same scattering effects as had been experimentally found ?

The AUTHOR, in reply, said the term "phase" was due to Rydberg, and was introduced before the Quantum-Theory was evolved. He did not know why the term had been selected. He was afraid he could not answer Prof. Howe's second question. These equations were fundamental to the Quantum-Theory, and to understand them fully one would have to know all about Quanta and the constitution of matter. With regard to the scattering of α particles, he thought it was possible to explain this as done by Hicks, if we take into account the magnetic action of the core. Of course, the electric action is also there and has to be taken into account as well. The results of experiments on scattering do not preclude the introduction of magnetic forces.

Communicated.—It appears that although Rydberg introduced the use of the symbol μ for the constant in his formula characteristic of a particular series, the term "phase" was first applied to the constant by Thiele (Astrophys. Journ. Vol. VI., p. 65, 1897). It is possible that the term was suggested by certain views as to the kinematical origin of spectra.

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